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Piecewise Smooth Spaces in Duality: Application to Blossoming

Marie-Laurence Mazure and Pierre-Jean Laurent

LMC-IMAG, Université Joseph Fourier, BP 53, 38041 Grenoble Cedex 9, France
E-mail: mazure@imag.fr, pjl@imag.fr

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Given an $(n+1)$-dimensional space $\mathscr{U}$ of piecewise smooth functions in which each basis has a non-vanishing Wronskian, and its dual space $\mathscr{U}^*$, a canonical bilinear form is defined on $\mathscr{U} \times \mathscr{U}^*$, which provides a simple characterization of a contact of order $r \leqslant n$. An intrinsic reproducing function is introduced, leading to Marsden-type identities. In the case of Chebyshev spaces connected with totally positive matrices, the bilinear form yields a general notion of blossom which can be extended to Chebyshev splines. © 1999 Academic Press

1. INTRODUCTION

The notion of *blossom* has recently brought a new insight both to the study of *geometric continuity* and to that of *Chebyshev splines* which were previously developed by different authors (see for instance S. Karlin and Z. Ziegler [7], L. L. Schumaker [26], P. E. Koch and T. Lyche [8, 9], and more particularly T. Lyche [12]). Initially introduced by L. Ramshaw [24, 25] for polynomial splines with parametric continuity conditions, the blossoming principle has been extended through two different approaches. The first one, used by H.-P. Seidel for geometrically continuous polynomial splines [27], then by R. Kulkarni, P.-J. Laurent, and M.-L. Mazure for *Q-splines* [10, 18], and later by H. Pottmann [22, 23, 29] for splines based on a given Chebyshev space (see also M.-L. Mazure and H. Pottmann [21], M.-L. Mazure [14, 16, 17], and M.-L. Mazure and P.-J. Laurent [20]), relies on geometric properties: the blossom is defined by means of intersections of convenient *osculating flats*. A second one, of a more algebraic nature, originates from the de Boor–Fix formula [3] and has first been developed by P. J. Barry in the case of splines with sections in arbitrary Chebyshev spaces [1]. On the other hand, several recent papers are devoted to *Marsden's identity*, *dual bases and blossoming* (see for instance P. J. Barry *et al.* [2], Y. Stefanus and R. N. Goldman [28],

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R. N. Goldman [5], and E. T. Y. Lee [11]). Extending the idea developed by M.-L. Mazure and P.-J. Laurent [19] for a single space, the present paper merges all these results and shows that in fact most of them are still valid in a more general framework, namely that of *piecewise smooth W-spaces.*

By a *W-space* we mean a finite dimensional space of C^∞ real valued functions defined on an interval, one basis of which has a nonvanishing Wronskian. Consider a subdivision $t_0 < t_1 < \cdots < t_q < t_{q+1}$ of the interval $I = [t_0, t_{q+1}]$, and denote by $\mathcal{S}$ the space of all functions defined on I whose restrictions to the subintervals $[t_l, t_{l+1}]$ belong to given $(n+1)$-dimensional W-spaces, and in which, for $l = 1, ..., q$, the left and right derivatives up to order $n_l (0 \leqslant n_l \leqslant n)$ at t_l are linked by a regular $(n_l + 1) \times (n_l + 1)$ connection matrix A_l. It is always possible to complement these matrices A_l into $(n+1) \times (n+1)$ regular ones such that the corresponding space $\mathcal{U}$ is included in $\mathcal{S}$. With such an $(n+1)$-dimensional *piecewise smooth W-space* $\mathcal{U}$ we can associate a *dual space* $\mathcal{U}^*$ of the same nature, a *canonical bilinear form* $[\cdot, \cdot]$ on $\mathcal{U} \times \mathcal{U}^*$, and a *reproducing function* E defined on $I \times I$. These latter two notions are connected by the relations

$$[U, E(x, \cdot)] = U(x), \qquad [E(\cdot, y), U^*] = U^*(y) \qquad x, y \in I. \tag{1.1}$$

The canonical bilinear form provides a simple and elegant characterization of the (left or right) contact of order $r \leqslant n$ between two elements $F, G \in \mathcal{U}$:

$$F^{(i)}(a^\varepsilon) = G^{(i)}(a^\varepsilon), \qquad i = 0, ..., r$$
$$\Leftrightarrow [F, \Psi^*] = [G, \Psi^*] \qquad \text{for all} \quad \Psi^* \in \mathcal{U}^* \quad \text{vanishing on } (a^\varepsilon)^{n-r}. \tag{1.2}$$

Furthermore, the reproducing function E leads to a Marsden-type identity: given a basis $S_{-n}, ..., S_m$ of $\mathcal{S}$, there exist $W^*_{-n}, ..., W^*_m \in \mathcal{U}^*$ such that

$$E(x, y) = \sum_{i=-n}^{m} S_i(x)\, W^*_i(y), \qquad x, y \in I. \tag{1.3}$$

Sections 2, 4, 5 of the present paper extend to the case of piecewise smooth spaces the theory developed in [19], in which we considered a single C^∞ space, and the results are applied to the study of splines in Sections 3 and 6. More precisely, Section 2 is devoted to the duality principle between piecewise smooth W-spaces, that is when two $(n+1)$-dimensional consecutive W-spaces are connected by an $(n+1) \times (n+1)$ matrix, while Section 3 deals with the notion of W-splines corresponding to connection matrices of lesser orders. In Section 4 we consider the particular case of a *piecewise smooth Chebyshev space*, i.e., a piecewise smooth W-space each section of which is a Chebyshev space. In Section 5, we suppose that such a space $\mathscr{U}$ contains the constant functions, and, most important, that the connections are defined by *totally positive* matrices applied on the classical differential operators related to the Chebyshev spaces involved. Then, using a result of P. J. Barry [1], a blossom can be defined in $\mathscr{U}$ by means of the canonical bilinear form:

$$f(x_1, ..., x_n) := [F, \Psi^*_{\mathscr{T}}], \tag{1.4}$$

where $\Psi^*_{\mathscr{T}}$ is the unique element of $\mathscr{U}^*$ which vanishes (with multiplicities) on $\mathscr{T} = (x_1, ..., x_n)$ and satisfies the normalization condition $[\mathbb{1}, \Psi^*_{\mathscr{T}}] = 1$. The characterization of contact stated in (1.2) enables the extension of the notion of blossom to the corresponding spline spaces: this is the subject of Section 6. Finally, in Section 7, we connect this algebraic approach with the geometrical one investigated in [17].

2. DUALITY BETWEEN PIECEWISE SMOOTH SPACES

Throughout this paper, we consider $q+2 (q \geqslant 0)$ fixed abscissae $t_0 < t_1 < t_2 \cdots < t_q < t_{q+1}$; we denote the whole interval by $I := [t_0, t_{q+1}]$ and the subintervals by $I_i := [t_i, t_{i+1}]$, $i = 0, ..., q$.

2.1. *Piecewise Smooth W-Spaces*

A function $U: I \to \mathbb{R}$ will be said to be *piecewise smooth on I* if $U \in C^0(I)$ and if, for $i = 0, ..., q$, its restriction to I_i belongs to $C^\infty(I_i)$. Moreover, in all the formulae to come, given $x \in I$, the notation x^ε is to be read either as x^+ or as x^- when x is one of the t_i's, $i = 1, ..., q$ (only x^+ if $x = t_0$, x^- if $x = t_{q+1}$); otherwise it can be replaced by x.

Given $n+1$ functions $U_0, ..., U_n$, assumed to be piecewise smooth on I, setting $\vec{U} := (U_0, ..., U_n)^T$, given $x^\varepsilon \in I$, let us introduce the following square matrix of order $n+1$:

$$\mathscr{W}_{\vec{U}}(x^\varepsilon) := \begin{pmatrix} U_0(x) & U_0'(x^\varepsilon) & \cdots & U_0^{(n)}(x^\varepsilon) \\ U_1(x) & U_1' x^\varepsilon) & \cdots & U_1^{(n)}(x^\varepsilon) \\ \vdots & \vdots & \ddots & \vdots \\ U_n(x) & U_n'(x^\varepsilon) & \cdots & U_n^{(n)}(x^\varepsilon) \end{pmatrix}. \tag{2.1}$$

The Wronskian (possibly left or right) of these functions is then defined by

$$W(U_0, ..., U_n)(x^\varepsilon) := \det \mathscr{W}_{\vec{U}}(x^\varepsilon). \tag{2.2}$$

DEFINITION 2.1. Let $\mathcal{U}$ be an $(n+1)$-dimensional space of piecewise smooth functions defined on I, and let $(U_0, ..., U_n)$ be a basis of $\mathcal{U}$. Then, $\mathcal{U}$ is said to be a *piecewise smooth W-space on I if*

$$W(U_0, ..., U_n)(x^\varepsilon) \neq 0 \qquad \text{for all} \quad x^\varepsilon \in I. \tag{2.3}$$

Clearly, this definition does not depend on the chosen basis. Actually, it means that a function $U \in \mathcal{U}$ is uniquely determined by knowing

$$\varDelta_n U(a^\varepsilon) := (U(a), U'(a^\varepsilon), ..., U^{(n)}(a^\varepsilon))^T \tag{2.4}$$

for any given $a^\varepsilon \in I$. Accordingly, for a given integer r, $-1 \leqslant r \leqslant n$, the set of all elements of $\mathcal{U}$ *vanishing on* $(a^\varepsilon)^{r+1}$ (i.e., which satisfy $U^{(j)}(a^\varepsilon) = 0$ for $j = 0, ..., r$) is an $(n-r)$-dimensional subspace of $\mathcal{U}$.

When $q = 0$, a piecewise smooth W-space on I will be said to be a *W-space on I*. A more explicit description of piecewise smooth W-spaces is given in the following proposition.

PROPOSITION 2.2. *Let $\mathcal{U}$ be a space of functions defined on I, and $\mathcal{U}_i$ its restriction to I_i, $i = 0, ..., q$. Then, $\mathcal{U}$ is an $(n+1)$-dimensional piecewise smooth W-space on I iff the following two conditions are satisfied:*

(i) *for $i = 0, ..., q$, $\mathcal{U}_i$ is a $(n+1)$-dimensional W-space on I_i,*

(ii) *there exist q regular square matrices of order $n+1$, say $M_1, ..., M_q$, with $(1, 0, ..., 0)$ as the first row, such that, for all $U \in \mathcal{U}$,*

$$\varDelta_n U(t_l^+) = M_l \cdot \varDelta_n U(t_l^-), \qquad l = 1, ..., q. \tag{2.5}$$

Proof. Let $(U_0, ..., U_n)$ be a basis of a given $(n+1)$-dimensional piecewise smooth W-space $\mathcal{U}$ on I. Then, for any $U = \sum_{i=0}^n \alpha_i U_i \in \mathcal{U}$, any $x^\varepsilon \in I$, we have

$$\varDelta_n U(x^\varepsilon) = \mathscr{W}_{\vec{U}}(x^\varepsilon)^T \cdot (\alpha_0, ..., \alpha_n)^T.$$

Applying this relation with $x^\varepsilon = t_l^-$ and $x^\varepsilon = t_l^+$ $(1 \leqslant l \leqslant q)$ leads to (2.5) with $M_l := \mathscr{W}_{\vec{U}}(t_l^+)^T \cdot \mathscr{W}_{\vec{U}}(t_l^-)^{-T}$. Since the two matrices $\mathscr{W}_{\vec{U}}(t_l^+)$ and $\mathscr{W}_{\vec{U}}(t_l^-)$ are regular, so is M_l. Moreover, the space $\mathcal{U}$ being contained in $C^0(I)$, it results from (2.5) that the first row of M_l is $(1, 0, ..., 0)$.

Conversely, assume (i) and (ii) to be satisfied. Due to the assumption on the first row of each M_l, $\mathcal{U}$ is a subspace of $C^0(I)$. Thus, in fact it only remains to prove that it is $(n+1)$-dimensional. Indeed, it is sufficient to check that the only element $U \in \mathcal{U}$ satisfying $\varDelta_n U(t_0) = 0$ is zero. The space $\mathcal{U}_0$ being an $(n+1)$-dimensional W-space on I_0, if $\varDelta_n U(t_0^+) = 0$, then the restriction of U to I_0 is necessarily zero. On account of (2.5), it will be possible to similarly prove, step by step, that its restriction to each interval I_i is also zero. ∎

For example, given q matrices $M_1, ..., M_q$ as in (ii), the space of all piecewise polynomial functions of degree less than or equal to n that satisfy (2.5) is an $(n+1)$-dimensional piecewise smooth W-space on I.

2.2. *The Dual Space of a Piecewise Smooth W-Space*

From now on, $\mathscr{U}$ will denote a given $(n+1)$-dimensional piecewise smooth W-space on I, and $(U_0, ..., U_n)$ a basis of $\mathscr{U}$. Then, for a given $x^\varepsilon \in I$, the $n+1$ following relations, in which $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^{n+1}$,

$$\langle \vec{U}^{(i)}(x^\varepsilon), \vec{U}^*(x^\varepsilon) \rangle = \begin{cases} 0 & \text{for} \quad i=0, ..., n-1, \\ 1 & \text{for} \quad i=n, \end{cases} \tag{2.6}$$

uniquely define a vector $\vec{U}^*(x^\varepsilon) = (U_0^*(x^\varepsilon), ..., U_n^*(x^\varepsilon))^T$, obtained by solving the linear system of order $n+1$

$$\mathscr{W}_{\vec{U}}(x^\varepsilon)^T \cdot \vec{U}^*(x^\varepsilon) = (0, ..., 0, 1)^T. \tag{2.7}$$

The system $(U_0^*, ..., U_n^*)$ so defined is called *the dual system* of $(U_0, ..., U_n)$ and the space $\mathscr{U}^* := \text{span}(U_0^*, ..., U_n^*)$ *the dual space* of $\mathscr{U}$. This space $\mathscr{U}^*$ can be shown not to depend on the basis $(U_0, ..., U_n)$.

PROPOSITION 2.3. *Let M_l, $l=1, ..., q$, be the connection matrices involved in the $(n+1)$-dimensional piecewise smooth W-space $\mathscr{U}$. Assume that the last column of each matrix M_l is equal to $(0, ..., 0, 1)^T$. Then, the dual space $\mathscr{U}^*$ of $\mathscr{U}$ is also an $(n+1)$-dimensional piecewise smooth W-space on I, and $\mathscr{U}^{**} = \mathscr{U}$.*

Proof. Observe first that, for $l=1, ..., q$, the fact that the last column of matrix M_l is equal to $(0, ..., 0, 1)^T$ is necessary and sufficient to ensure the equality $\vec{U}^*(t_l^+) = \vec{U}^*(t_l^-)$. Thus, the vector valued function $\vec{U}^*$ is well-defined on I. Moreover, as it is obtained by solving (2.7), it is C^∞ on each subinterval, hence it is piecewise smooth on I.

Furthermore, by using a recursive argument, differentiating (2.6) on each subinterval leads to the following relations for all $x^\varepsilon \in I$:

$$\langle \vec{U}^{(i)}(x^\varepsilon), \vec{U}^{*(j)}(x^\varepsilon) \rangle = \begin{cases} 0 & \text{for} \quad i+j \leqslant n-1, \\ (-1)^j & \text{for} \quad i+j=n. \end{cases} \tag{2.8}$$

Let us consider the square matrix $\mathscr{L}(x^\varepsilon)$ of order $n+1$ defined by

$$(\mathscr{L}(x^\varepsilon))_{i,j} := \langle \vec{U}^{(i)}(x^\varepsilon), \vec{U}^{*(j)}(x^\varepsilon) \rangle, \qquad x^\varepsilon \in I, \tag{2.9}$$

or, equivalently, by

$$\mathscr{Z}(x^\varepsilon) := \mathscr{W}_{\vec{U}}(x^\varepsilon)^T \cdot \mathscr{W}_{\vec{U}*}(x^\varepsilon). \tag{2.10}$$

Then, relations (2.8) mean that $\mathscr{Z}(x^\varepsilon)$ has the structure

$$\mathscr{Z}(x^\varepsilon) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & (-1)^n \\ \vdots & \vdots & & (-1)^{n-1} & \cdot \\ & & \cdot^{\cdot^{\cdot}} & \cdot & \cdot \\ 0 & -1 & & & \\ 1 & \cdot & & \cdot & \cdot \end{pmatrix}. \tag{2.11}$$

Since $\mathscr{Z}(x^\varepsilon)$ is regular for any $x^\varepsilon \in I$, so is $\mathscr{W}_{\vec{U}*}(x^\varepsilon)$, which means that the Wronskian $W(U_0^*, ..., U_n^*)(x^\varepsilon)$ never vanishes on I. Hence, the space $\mathscr{U}^*$ is also an $(n+1)$-dimensional piecewise smooth W-space on I.

Actually, the case $i=0$ in (2.8) shows that the dual system of $(U_0^*, ..., U_n^*)$ is equal to $(-1)^n (U_0, ..., U_n)$. As an immediate consequence, the dual space of the dual space $\mathscr{U}^*$ is the initial space $\mathscr{U}$. ∎

Let us observe that, on account of (2.10), for any function U^* of the dual space $\mathscr{U}^*$, the connection at t_l will be given by

$$\Delta_n U^*(t_l^+) = M_l^* \cdot \Delta_n U^*(t_l^-), \qquad l = 1, ..., q, \tag{2.12}$$

where M_l^* is the following regular square matrix of order $n+1$:

$$M_l^* := \mathscr{Z}(t_l^+)^T \cdot M_l^{-T} \cdot \mathscr{Z}(t_l^-)^{-T}. \tag{2.13}$$

2.3. *Reproducing Function and Canonical Bilinear Form*

Let us denote by $\mathscr{M}$ the set of all regular matrices of order $n+1$ which have $(1, 0, ..., 0)$ as the first row and $(0, ..., 0, 1)^T$ as the last column. From now on, we shall always suppose that the connection matrices $M_1, ..., M_q$ all belong to $\mathscr{M}$. Then, given a basis $(U_0, ..., U_n)$ of $\mathscr{U}$ and its dual system $(U_0^*, ..., U_n^*)$, we can consider the function $E: I \times I \to \mathbb{R}$ defined by

$$E(x, y) := \langle \vec{U}(x), \vec{U}^*(y) \rangle = \sum_{i=0}^n U_i(x)\, U_i^*(y), \qquad x, y \in I. \tag{2.14}$$

LEMMA 2.4. *There exists a unique bilinear form $[\cdot, \cdot]$ on $\mathscr{U} \times \mathscr{U}^*$ such that*

$$[U, E(x, \cdot)] = U(x) \qquad \textit{for all} \quad U \in \mathscr{U} \quad \textit{and all} \quad x \in I. \tag{2.15}$$

It is the nondegenerate bilinear form characterized by

$$[U_i, U_j^*] = \delta_{ij} \qquad \textit{for all} \quad i, j = 0, ..., n. \tag{2.16}$$

Proof. Let us first denote by $[\cdot, \cdot]$ the bilinear form defined on $\mathscr{U} \times \mathscr{U}^*$ by (2.16). Then, any $U \in \mathscr{U}$, and any $U^* \in \mathscr{U}^*$ can be written as

$$U = \sum_{i=0}^{n} [U, U_i^*] U_i, \qquad U^* = \sum_{i=0}^{n} [U_i, U^*] U_i^*. \tag{2.17}$$

For $x \in I$, $E(x, \cdot)$ being the element of $\mathscr{U}^*$ defined by $E(x, \cdot) = \sum_{i=0}^{n} U_i(x) U_i^*$, we have $U_i(x) = [U_i, E(x, \cdot)]$, $i = 0, ..., n$, which yields (2.15) by linearity.

Since the functions $(U_0, ..., U_n)$ are linearly independent, one can select $x_0, ..., x_n \in I$ so that the square matrix A of order $n+1$ defined by $A_{i,j} := U_j(x_i)$, $i, j = 0, ..., n$, is regular. The equality

$$(E(x_0, \cdot), ..., E(x_n, \cdot))^T = A \cdot (U_0^*, ..., U_n^*)^T$$

proves that $(E(x_0, \cdot), ..., E(x_n, \cdot))$ is a basis of $\mathscr{U}^*$. Similarly, it is possible to select $n+1$ points $y_0, ..., y_n$ in I such that $(E(\cdot, y_0), ..., E(\cdot, y_n))$ form a basis of $\mathscr{U}$.

A bilinear form $[\cdot, \cdot]$ on $\mathscr{U} \times \mathscr{U}^*$ is thus completely determined as soon as the quantities $[E(\cdot, y_j), E(x_i, \cdot)]$, $i, j = 0, ..., n$, are known. If $[\cdot, \cdot]$ is assumed to satisfy (2.15), we necessarily have

$$[E(\cdot, y_j), E(x_i, \cdot)] = E(x_i, y_j), \qquad i, j = 0, ..., n, \tag{2.18}$$

which proves the unicity of such a bilinear form. ∎

LEMMA 2.5. *Given $x^\varepsilon, y^{\varepsilon'} \in I$, $E(\cdot, y)$ is the only element $U \in \mathscr{U}$ such that*

$$\Delta_n U(y^{\varepsilon'}) = (0, ..., 0, 1)^T, \tag{2.19}$$

while $E(x, \cdot)$ is the only element $U^* \in \mathscr{U}^*$ such that

$$\Delta_n U^*(x^\varepsilon) = (0, ..., 0, (-1)^n)^T. \tag{2.20}$$

Proof. From (2.14), we can calculate the (possibly left or right) partial derivatives of E as

$$\partial_{1^i 2^j} E(x^\varepsilon, y^{\varepsilon'}) = \langle \vec{U}^{(i)}(x^\varepsilon), \vec{U}^{*(j)}(y^{\varepsilon'}) \rangle, \qquad x^\varepsilon, y^{\varepsilon'} \in I. \tag{2.21}$$

Consequently, relations (2.6) mean that, for a given $y^{\varepsilon'} \in I$, the function $U := E(\cdot, y)$ satisfies (2.19). Moreover, U being an element of the piecewise smooth W-space $\mathscr{U}$, it is completely determined by (2.19).

Similarly, (2.20) can be derived from (2.21) and (2.8). ∎

Let us mention that N. Dyn and A. Ron have already introduced the function defined by (2.19) when $\mathscr{U}$ is a W-space [4].

THEOREM 2.6. *The function E, the bilinear form $[\cdot, \cdot]$ defined by* (2.16) *and the matrix $\mathscr{L}$ defined by* (2.10) *are intrinsic, in the sense that they depend only on the space $\mathscr{U}$, not on the chosen basis* $(U_0, ..., U_n)$.

Proof. The fact that E is intrinsic is a straightforward consequence of the previous lemma. The bilinear form defined in (2.16) being associated with E, it is also intrinsic. Finally, the same holds for $\mathscr{L}$, on account of the equality

$$(\mathscr{L}(x^\varepsilon))_{ij} = \partial_{1^i 2^j} E(x^\varepsilon, x^\varepsilon) \qquad \text{for all} \quad x^\varepsilon \in I. \quad \blacksquare \tag{2.22}$$

DEFINITION 2.7. The function E and the bilinear form $[\cdot, \cdot]$ will be called *the reproducing function* and *the canonical bilinear form* associated with the piecewise smooth W-space $\mathscr{U}$, respectively.

Let us observe that, symmetrically, the canonical bilinear form also satisfies the reproducing property

$$[E(\cdot, y), U^*] = U^*(y) \qquad \text{for all} \quad U^* \in \mathscr{U}^* \quad \text{and all} \quad y \in I. \tag{2.23}$$

Remark 2.8. In the particular case of an $(n+1)$-dimensional piecewise polynomial W-space on I, it follows from (2.19) that

$$E(x, y) = (x-y)^n/n! \tag{2.24}$$

as soon as x and y belong to the same subinterval I_i. Therefore, (2.22) shows that, for all $x^\varepsilon \in I$,

$$\mathscr{L}(x^\varepsilon) = \mathscr{R}, \tag{2.25}$$

where $\mathscr{R}$ stands for the antidiagonal matrix of order $n+1$ such that $\mathscr{R}_{i,j} = (-1)^j$ for $0 \leqslant i \leqslant n$, $i + j = n$. The existence of this particular function E has been pointed out by P. J. Barry *et al.* by using a different argument [2].

We conclude this subsection with the following result.

PROPOSITION 2.9. *Let $U_0, ..., U_n$ be $n+1$ elements of $\mathscr{U}$, and $V_0^*, ..., V_n^*$ be $n+1$ elements of $\mathscr{U}^*$. Then the following three properties are equivalent*

(i) $E(x, y) = \sum_{i=0}^n U_i(x)\, V_i^*(y)$ *for all* $x, y \in I$,

(ii) $[U_i, V_j^*] = \delta_{ij}$, $i, j = 0, ..., n$,

(iii) $(U_0, ..., U_n)$ *is a basis of $\mathscr{U}$ and* $(V_0^*, ..., V_n^*)$ *is its dual system.*

Proof. Taking into account both Theorem 2.6 and (2.16), the equivalence between (ii) and (iii) is clear.

Suppose that (i) is valid. Then, given $U \in \mathscr{U}$, from $U(x) = [U, E(x, \cdot)]$ for all $x \in I$, we obtain

$$U = \sum_{i=0}^{n} [U, V_i^*] U_i,$$

which proves that $(U_0, ..., U_n)$ is a basis of $\mathscr{U}$. As usual, let us denote by $(U_0^*, ..., U_n^*)$ its dual basis. By (2.14) and Theorem 2.6, we have, for any $y \in I$, $E(\cdot, y) = \sum_{i=0}^{n} U_i^*(y) U_i$. On account of (i), this proves the equality $\sum_{i=0}^{n} U_i^*(y) U_i = \sum_{i=0}^{n} V_i^*(y) U_i$, hence $V_i^* = U_i^*$. The converse part is obvious. ∎

2.4. *Canonical Bilinear Form and Contact*

In this subsection, it will be proved that the canonical bilinear form provides an easy characterization of the (possibly left or right) contact of order $r \leqslant n$ between two elements of $\mathscr{U}$.

PROPOSITION 2.10. *For any $U \in \mathscr{U}$ and any $U^* \in \mathscr{U}^*$, we have, for any $a^\varepsilon \in I$,*

$$[U, U^*] = \Delta_n U(a^\varepsilon)^T \cdot \mathscr{L}(a^\varepsilon)^{-T} \cdot \Delta_n U^*(a^\varepsilon). \tag{2.26}$$

Proof. Let $(U_0, ..., U_n)$ be a basis of $\mathscr{U}$ and $(U_0^*, ..., U_n^*)$ be its dual system. Given $U = \sum_{i=0}^{n} \alpha_i U_i \in \mathscr{U}$ and $U^* = \sum_{i=0}^{n} \alpha_i^* U_i^* \in \mathscr{U}^*$, we have

$$\begin{aligned} \Delta_n U(a^\varepsilon) &= \mathscr{W}_{\vec{U}}(a^\varepsilon)^T \cdot (\alpha_0, ..., \alpha_n)^T, \\ \Delta_n U^*(a^\varepsilon) &= \mathscr{W}_{\vec{U}^*}(a^\varepsilon)^T \cdot (\alpha_0^*, ..., \alpha_n^*)^T. \end{aligned} \tag{2.27}$$

Since, by (2.16), $[U, U^*] = (\alpha_0, ..., \alpha_n) \cdot (\alpha_0^*, ..., \alpha_n^*)^T$, relations (2.27) yield

$$[U, U^*] = \Delta_n U(a^\varepsilon)^T \cdot \mathscr{W}_{\vec{U}}(a^\varepsilon)^{-1} \cdot \mathscr{W}_{\vec{U}^*}(a^\varepsilon)^{-T} \cdot \Delta_n U^*(a^\varepsilon). \tag{2.28}$$

Now, according to Theorem 2.6, this quantity is independent of the basis. For instance, we can choose the basis $(U_0, ..., U_n)$ of $\mathscr{U}$ characterized by $U_i^{(j)}(a^\varepsilon) = \delta_{ij}$, $i, j = 0, ..., n$, that is to say, by

$$\mathscr{W}_{\vec{U}}(a^\varepsilon) = \mathscr{I}_{n+1}. \tag{2.29}$$

Then, by (2.10) the dual system of this basis satisfies $\mathscr{W}_{\vec{U}^*}(a^\varepsilon) = \mathscr{L}(a^\varepsilon)$. Consequently, for this particular choice, (2.28) reduces to (2.26). ∎

Remark 2.11. In the piecewise polynomial case, using (2.25), formula (2.26) leads to

$$[U, U^*] = \Delta_n U(a^\varepsilon)^T \cdot \mathscr{R} \cdot \Delta_n U^*(a^\varepsilon)$$

since $\mathscr{R}^{-T} = \mathscr{R}$, i.e.,

$$[U, U^*] = \sum_{k=0}^{n} U^{(k)}(a^\varepsilon)(-1)^{n-k} U^{*(n-k)}(a^\varepsilon). \tag{2.30}$$

Note that the right hand sides of (2.26) and (2.30) are necessarily independent of $a^\varepsilon \in I$.

COROLLARY 2.12. *Suppose that $a^\varepsilon \in I$ and let r be an integer such that $0 \leqslant r \leqslant n$. Then, a function $\Psi^* \in \mathscr{U}^*$ vanishes on $(a^\varepsilon)^{n-r}$ (i.e., satisfies $\Psi^{*(j)}(a^\varepsilon) = 0$ for $j = 0, ..., n-r-1$) iff there exist r real numbers $\lambda_0, ..., \lambda_{r-1}$ such that, for all $U \in \mathscr{U}$,*

$$\begin{aligned}[U, \Psi^*] = {} & \lambda_0 U(a) + \lambda_1 U'(a^\varepsilon) + \cdots + \lambda_{r-1} U^{(r-1)}(a^\varepsilon) \\ & + (-1)^{n-r} \Psi^{*(n-r)}(a^\varepsilon)\, U^{(r)}(a^\varepsilon).\end{aligned} \tag{2.31}$$

Proof. Given $\Psi^* \in \mathscr{U}^*$, formula (2.26) proves that

$$[U, \Psi^*] = \sum_{i=0}^{n} \lambda_i U^{(i)}(a^\varepsilon) \qquad \text{for all} \quad U \in \mathscr{U}, \tag{2.32}$$

where

$$(\lambda_0, ..., \lambda_n)^T := \mathscr{L}(a^\varepsilon)^{-T} \cdot \varDelta_n \Psi^*(a^\varepsilon). \tag{2.33}$$

Now, Ψ^* vanishes on $(a^\varepsilon)^{n-r}$ iff the $(n-r)$ first components of $\varDelta_n \Psi^*(a^\varepsilon)$ are equal to 0. Taking into account the regularity and the structure of $\mathscr{L}(a^\varepsilon)^{-T}$, this occurs iff $\lambda_{r+1} = \cdots = \lambda_n = 0$. Moreover, in that case, (2.33) yields $\lambda_r = (-1)^{n-r} \Psi^{*(n-r)}(a^\varepsilon)$, which completes the proof. ▮

For any subset $\mathscr{A}$ of $\mathscr{U}$, the subspace of $\mathscr{U}^*$ which is orthogonal to $\mathscr{A}$ with respect to the canonical bilinear form will be denoted by $\mathscr{A}^\circ$, namely

$$\mathscr{A}^\circ := \{U^* \in \mathscr{U}^* \mid [U, U^*] = 0,\ \forall U \in \mathscr{A}\}. \tag{2.34}$$

One can define similarly $\mathscr{A}^{*\circ}$ for any subset $\mathscr{A}^*$ of $\mathscr{U}^*$.

COROLLARY 2.13. *Given an integer r, $-1 \leqslant r \leqslant n$, and $a^\varepsilon \in I$,*

$$\{U \in \mathscr{U} \mid U \textit{ vanishes on } (a^\varepsilon)^{r+1}\}^\circ = \{U^* \in \mathscr{U}^* \mid U^* \textit{ vanishes on } (a^\varepsilon)^{n-r}\}. \tag{2.35}$$

Proof. Let us introduce the following subspace of $\mathscr{U}$:

$$\mathscr{A} := \{U \in \mathscr{U} \mid U \text{ vanishes on } (a^\varepsilon)^{r+1}\}, \tag{2.36}$$

which is $(n-r)$-dimensional. It results from Corollary 2.12 that

$$\{U^* \in \mathscr{U}^* \mid U^* \text{ vanishes on } (a^\varepsilon)^{n-r}\} \subset \mathscr{A}^\circ. \tag{2.37}$$

The canonical bilinear form being nondegenerate, $\mathscr{A}^\circ$ is of dimension $r+1$, i.e., of the same dimension as the left hand side of (2.37), which leads to (2.35). ∎

Two functions $F, G \in \mathscr{U}$ will be asid to have a *contact of order* $r \leqslant n$ at $a^\varepsilon \in I$ if $F^{(i)}(a^\varepsilon) = G^{(i)}(a^\varepsilon)$ for all $i = 0, ..., r$. The previous corollary provides the following characterization of this notion of contact.

THEOREM 2.14. *Two functions $F, G \in \mathscr{U}$ have a contact of order $r \leqslant n$ at $a^\varepsilon \in I$ iff $[F, \Psi^*] = [G, \Psi^*]$ for every function $\Psi^* \in \mathscr{U}^*$ that vanishes on $(a^\varepsilon)^{n-r}$.*

Proof. Clearly, F and G have a contact of order r at a^ε iff $U := F - G$ belongs to the subspace $\mathscr{A}$ defined in (2.36). According to Corollary 2.13, U belongs to $\mathscr{A} = \mathscr{A}^{\circ\circ}$ iff $[U, \Psi^*] = 0$ for all Ψ^* vanishing on $(a^\varepsilon)^{n-r}$. ∎

3. W-SPLINES, CONTACT AND MARSDEN-TYPE IDENTITIES

In the whole section, for $l = 1, ..., q$, A_l denotes a regular square matrix of order $n_l + 1$ $(0 \leqslant n_l \leqslant n)$, the first row of which is $(1, 0, ..., 0)$. Moreover, in case $n_l = n$, we additionally require the last column of A_l to be $(0, ..., 0, 1)^T$ (which actually means, in that case, that $A_l \in \mathscr{M}$).

3.1. *W-Splines*

For all $i = 0, ..., q$, $\mathscr{U}_i$ is assumed to be a W-space on I_i. Let us denote by $\mathscr{S}$ the space of all functions $S: I \to \mathbb{R}$ the restriction of which to I_i belongs to $\mathscr{U}_i$, for $i = 0, ..., q$, and which satisfy

$$\Delta_{n_l} S(t_l^+) = A_l \cdot \Delta_{n_l} S(t_l^-), \qquad l = 1, ..., q. \tag{3.1}$$

Let us define the multiplicity m_l at t_l by $m_l := n - n_l$ for $l = 1, ..., q$. Then, since each $\mathscr{U}_i$ is a W-space on I_i, the dimension of $\mathscr{S}$ is equal to $n + m + 1$, with $m := \sum_{l=1}^q m_l$.

DEFINITION 3.1. The space $\mathscr{S}$ defined above is called *the W-spline space* associated with both the sequence $\mathscr{U}_0, ..., \mathscr{U}_q$ of *W*-spaces and the sequence $A_1, ..., A_q$ of matrices.

For each $l=1, ..., q$, let us complement the matrix A_l so as to create a matrix $M_l=(m_{ij}^l)_{ij=0,...,n}$ belonging to $\mathcal{M}$, with the additional requirement that

$$m_{ij}^l=0, \qquad i=0, ..., n_l, \qquad j=n_l+1, ..., n. \tag{3.2}$$

Now, consider the piecewise smooth W-space $\mathcal{U}$ constructed with these matrices M_l as in Section 2, and $\mathcal{U}^*$ its dual space. Clearly, due to (3.2), we have

$$\mathcal{U} \subset \mathcal{S}. \tag{3.3}$$

On account of (3.2), equality (2.13) proves that, in the dual space $\mathcal{U}^*$, the connection matrices $M_l^*=(m_{ij}^{*l})_{ij=0,...,n}$ satisfy

$$m_{ij}^{*l}=0, \qquad i=0, ..., m_l-1, \qquad j=m_l, ..., n. \tag{3.4}$$

The structures of M_l and M_l^* are illustrated below for $n=4$ and $n_l=2$.

$$M_l=\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 & 0 \\ \hline \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right] \qquad M_l^*=\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & 0 \\ \hline \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 1 \end{array}\right]$$

In other words, (3.4) means that M_l^* has a block lower triangular structure. Therefore, for $l=1, ..., q$, there exists a square matrix A_l^* of order m_l (with $(1, 0, ..., 0)$ as its first row) such that

$$\Delta_{m_l-1} U^*(t_l^+)=A_l^* \cdot \Delta_{m_l-1} U^*(t_l^-) \qquad \text{for all} \quad U^* \in \mathcal{U}^*. \tag{3.5}$$

Moreover, M_l^* being regular, so is A_l^*. Accordingly, $U^* \in \mathcal{U}^*$ vanishes on $(t_l^+)^{m_l}$ iff it vanishes on $(t_l^-)^{m_l}$. If so, we shall simply say that it vanishes on $(t_l)^{m_l}$.

On the other hand, given $S \in \mathcal{S}$ and $i \in \{0, ..., q\}$, since $\mathcal{U}$ is a piecewise smooth W-space, there exists a unique element of $\mathcal{U}$ which coincides with S on I_i.

Proposition 3.2. *Given a spline function $S \in \mathcal{S}$ and $i \in \{0, ..., q\}$, let us denote by $\tilde{S}_i$ the unique element of $\mathcal{U}$ which satisfies $\tilde{S}_{i\,|\,I_i}=S_{|\,I_i}$. Then, for $l=1, ..., q$,*

$$[\tilde{S}_l, \Psi^*]=[\tilde{S}_{l-1}, \Psi^*]$$

for all $\Psi^* \in \mathcal{U}^*$ which vanishes on $(t_l^{m_l})$.

Proof. Since $S \in \mathscr{S}$, it satisfies (3.1), which can also be written

$$\Delta_{n_l} \tilde{S}_l(t_l^+) = A_l \cdot \Delta_{n_l} \tilde{S}_{l-1}(t_l^-), \tag{3.6}$$

due to the fact that $\tilde{S}_{l-1}$ and $\tilde{S}_l$ coincide with S on I_{l-1} and I_l respectively. On the other hand, functions $\tilde{S}_{l-1}$ and $\tilde{S}_l$ belong to $\mathscr{U} \subset \mathscr{S}$. Accordingly, we have

$$\Delta_{n_l} \tilde{S}_{l-1}(t_l^+) = A_l \cdot \Delta_{n_l} \tilde{S}_{l-1}(t_l^-), \qquad \Delta_{n_l} \tilde{S}_l(t_l^+) = A_l \cdot \Delta_{n_l} \tilde{S}_l(t_l^-). \tag{3.7}$$

Comparing (3.6) and (3.7), we obtain

$$\Delta_{n_l} \tilde{S}_l(t_l^+) = \Delta_{n_l} \tilde{S}_{l-1}(t_l^+), \qquad \Delta_{n_l} \tilde{S}_l(t_l^-) = \Delta_{n_l} \tilde{S}_{l-1}(t_l^-). \tag{3.8}$$

Hence, the announced result is a consequence of Theorem 2.14. ∎

3.2. *Admissible Tuples*

With the sequence of multiplicities $m_1, ..., m_q$ (and, in addition, for the end points $m_0 = m_{q+1} := n+1$), let us associate the corresponding knot vector,

$$T := (t_0^{m_0} t_1^{m_1} \cdots t_q^{m_q} t_{q+1}^{m_{q+1}}), \tag{3.9}$$

where the notation $t_i^{m_i}$ means that t_i is repeated m_i times. Moreover, associated with an arbitrary p-tuple $\mathscr{T} \in I^p$, we consider the p-tuple $\mathscr{T}^{ord}$ composed of the same elements as $\mathscr{T}$, but arranged in ascending order. Following the multiplicative notation introduced above, it can be written $\mathscr{T}^{ord} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$, with positive integers μ_i and $\tau_1 < \tau_2 \cdots < \tau_r$.

Definition 3.3. Let $\mathscr{T}$ be an element of I^p, $p \leqslant n+1$, with $\mathscr{T}^{ord} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$. Then, $\mathscr{T}$ will be said to be *admissible* with respect to the knot vector T if every t_i $(1 \leqslant i \leqslant q)$ belonging to $\mathrm{ri}[\tau_1, \tau_r]$ is repeated at least m_i times in $\mathscr{T}$.

The notation $\mathrm{ri}[\alpha, \beta]$ stands for the relative interior of the interval $[\alpha, \beta]$, i.e., $]\alpha, \beta[$ when $\alpha < \beta$ and $\{\alpha\}$ when $\alpha = \beta$. Therefore, for $i = 0$ or $i = q+1$, the p-tuple (t_i^p) is admissible for all $p \leqslant n+1$, whereas for $1 \leqslant i \leqslant q$, (t_i^p) is admissible iff $p \geqslant m_i$. In particular, since the multiplicity at each interior knot is supposed to be less than or equal to n, the k-tuple (t^n) is admissible whatever the point $t \in I$ may be.

Definition 3.4. If $\mathscr{T}$ is an admissible p-tuple, $p \leqslant n$, its *domain* is defined as

$$\mathscr{D}(\mathscr{T}) := \{t \in I / (t, \mathscr{T}) \text{ is admissible}\}. \tag{3.10}$$

LEMMA 3.5. *Let $\mathscr{T}$ be an admissible p-tuple, $p \leqslant n$. Then, $\mathscr{D}(\mathscr{T})$ is a union of consecutive intervals I_i, i.e.,*

$$\mathscr{D}(\mathscr{T}) = \bigcup_{i \in \mathscr{J}(\mathscr{T})} I_i, \tag{3.11}$$

where $\mathscr{J}(\mathscr{T})$ is a nonempty subset of consecutive integers.

Proof. Let $\mathscr{N}(\mathscr{T})$ denote the set of all integers i, $1 \leqslant i \leqslant q$, such that $m_i > 0$ and that t_i appears at least m_i times in $\mathscr{T}$. Let us set $\mathscr{T}^{ord} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$. Two possibilities must be examined.

(1) $\mathscr{N}(\mathscr{T}) = \varnothing$. Let i be the greatest integer such that $m_i > 0$ and $t_i \leqslant \tau_1$, and $i+s$ the smallest integer such that $m_{i+s} > 0$ and $t_{i+s} \geqslant \tau_r$. On account of both the admissibility of $\mathscr{T}$ and the fact that $\mathscr{N}(\mathscr{T}) = \varnothing$, we have $s \geqslant 1$ except when $\mathscr{T} = (t_0^p)$ or $\mathscr{T} = (t_{q+1}^p)$. If for instance $\mathscr{T} = (t_0^p)$, then clearly $\mathscr{D}(\mathscr{T}) = [t_0, t_k]$, where k is the smallest positive integer such that $m_k > 0$. On the other hand, if $s \geqslant 1$ one can easily check that $\mathscr{D}(\mathscr{T}) = [t_i, t_{i+s}]$, i.e., $\mathscr{J}(\mathscr{T}) = \{i, ..., i+s-1\}$.

(2) $\mathscr{N}(\mathscr{T}) \neq \varnothing$. In that case, we have $\mathscr{D}(\mathscr{T}) = [t_i, t_{i+s}]$, where i denotes the greatest integer such that $m_i > 0$ and $t_i < \operatorname{Min} \mathscr{N}(\mathscr{T})$, and $i+s$ the smallest integer such that $m_{i+s} > 0$ and $t_{i+s} > \operatorname{Max} \mathscr{N}(\mathscr{T})$. ∎

PROPOSITION 3.6. *Given an admissible n-tuple $\mathscr{T} \in I^n$, with $\mathscr{T}^{ord} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$, and $\varepsilon_1, ..., \varepsilon_r$ such that $\tau_i^{\varepsilon_i} \in I$, let Ψ^* be an element of $\mathscr{U}^*$ which vanishes on each $(\tau_i^{\varepsilon_i})^{\mu_i}$, $i = 1, ..., r$. Then, with the same notations as in Proposition 3.2, all the quantities $[\tilde{S}_l, \Psi^*]$, $l \in \mathscr{J}(\mathscr{T})$, are equal.*

Proof. Suppose that $\mathscr{J}(\mathscr{T})$ contains two consecutive integers $l-1$ and l, $1 \leqslant l \leqslant q$. Then, $\mathscr{T}$ being admissible, t_l appears at least m_l times in $\mathscr{T}$. So, there exists an integer $i_0 \in \{1, ..., r\}$ such that $\tau_{i_0} = t_l$ and we have $\mu_{i_0} \geqslant m_l$. Since Ψ^* is assumed to vanish on $(\tau_{i_0}^{\varepsilon_{i_0}})^{\mu_{i_0}}$, it vanishes on $t_l^{m_l}$. On account of Proposition 3.2, this ensures the equality

$$[\tilde{S}_{l-1}, \Psi^*] = [\tilde{S}_l, \Psi^*]. \quad ∎ \tag{3.12}$$

3.3. *Marsden-Type Identities*

THEOREM 3.7. *Let $(S_{-n}, ..., S_m)$ be a basis of $\mathscr{S}$. Then, there exist $n+m+1$ functions $W^*_{-n}, ..., W^*_m \in \mathscr{U}^*$ such that*

$$E(x, y) = \sum_{i=-n}^{m} S_i(x)\, W_i^*(y) \qquad \textit{for all} \quad x, y \in I, \tag{3.13}$$

where E is the reproducing function associated with $\mathscr{U}$. These functions $W^*_{-n}, ..., W^*_m$ span the space $\mathscr{U}^*$.

Proof. For a given $y \in I$, since function $E(\cdot, y)$ belongs to $\mathscr{U}$, it also belongs to $\mathscr{S}$. Let us denote its coordinates in the basis $(S_{-n}, ..., S_m)$ by $W^*_{-n}(y), ..., W^*_m(y)$:

$$E(\cdot, y) = \sum_{i=-n}^{m} W^*_i(y) S_i. \tag{3.14}$$

By selecting $n+m+1$ points $x_{-n}, ..., x_m \in I$ such that $\det(S_i(x_j))_{i, j=-n, ..., m} \neq 0$, we can deduce from (3.14) that each W^*_i is a linear combination of $E(x_j, \cdot)$, $j=-n, ..., m$. Hence, W^*_i is an element of $\mathscr{U}^*$.

On the other hand, since E is the reproducing function of $\mathscr{U}$, the space $\mathscr{U}^*$ is spanned by all functions $E(x, \cdot)$, $x \in I$. Consequenly, on account of (3.13), it is also spanned by $(W^*_{-n}, ..., W^*_m)$. ∎

Remark 3.8. Formula (3.13) is a Marsden-type identity. It has been obtained from the reproducing function associated with the $(n+1)$-dimensional piecewise smooth W-space $\mathscr{U} \subset \mathscr{S}$. But this space $\mathscr{U}$ is determined by the choice of the matrices M_l. So, different ways of complementing the A_i's into elements of $\mathscr{M}$ (with the requirement (3.2)) will lead to different Marsden-type identities related to the same W-spline space $\mathscr{S}$ and the same basis $S_{-n}, ..., S_m$ of $\mathscr{S}$.

More generally, whatever its dimension, each piecewise smooth W-space $\mathscr{U}$ contained in $\mathscr{S}$ provides such an identity. As a simple example, let us consider the space $\mathscr{S}$ of C^{n-1} polynomial splines of degree n defined on I. Then, for a given $v \leqslant n$, the space $\mathscr{U} := \mathscr{P}_v$ of all polynomials of degree less than or equal to v defined on I is contained in $\mathscr{S}$. Here $m=q$ and one corresponding Marsden-type identity can be written

$$\frac{(x-y)^v}{v!} \equiv \sum_{i=-n}^{q} \mathscr{N}_i(x)\, W^*_i(y), \qquad x, y \in I, \tag{3.15}$$

where $\mathscr{N}_i$, $i=-n, ..., q$, denotes the classical B-spline basis of degree n, each W^*_i being a polynomial of degree less than or equal to v. It is now well-known that the B-spline basis is the dual basis of the linear forms on $\mathscr{S}$

$$\phi_i : S \in \mathscr{S} \mapsto s(t_{i+1}, ... \, t_{i+n}), \qquad i=-n, ..., q,$$

where s is the blossom of S. Therefore, $W^*_i(y)$ is the value of the blossom of function $(\cdot - y)^v/v!$ at $(t_{i+1}, ..., t_{i+n})$, i.e.,

$$W^*_i(y) = \frac{1}{v!\binom{n}{v}} \sum_{\substack{|K|=v \\ K \subset \{i+1, ..., i+n\}}} \prod_{j \in K} (t_j - y). \tag{3.16}$$

The classical Marsden's identity corresponds to the case $v=n$.

4. PIECEWISE SMOOTH EC-SPACES

A piecewise smooth W-space $\mathscr{U}$ on I is said to be a *piecewise smooth EC space* on I if, for $i=0, ..., q$, its restriction $\mathscr{U}_i$ to I_i is an extended Chebyshev space (shortly, EC) space on I_i.

4.1. *EC-Spaces and Weight Functions*

In this subsection, we shall give a compact presentation of the necessary tools on Chebyshev spaces. For the proofs and more details, see for instance [6, 15, 26].

Let J be a closed bounded real interval and $\mathscr{E}$ be an $(n+1)$-dimensional subspace of $C^\infty(J)$. Then, $\mathscr{E}$ is said to be *an extended Chebyshev space on* J if any nonzero element of $\mathscr{E}$ has at most n zeros (counted with multiplicities) on J, or, equivalently, if a given basis $(E_0, ..., E_n)$ of $\mathscr{E}$ satisfies

$$\begin{vmatrix} E_0(\tau_0) & \cdots & E_0^{(\mu_0-1)}(\tau_0) & E_0(\tau_1) & \cdots & E_0^{(\mu_r-1)}(\tau_r) \\ E_1(\tau_0) & & E_1^{(\mu_0-1)}(\tau_0) & E_1(\tau_1) & \cdots & E_1^{(\mu_r-1)}(\tau_r) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ E_n(\tau_0) & \cdots & E_n^{(\mu_0-1)}(\tau_0) & E_n(\tau_1) & \cdots & E_n^{(\mu_r-1)}(\tau_r) \end{vmatrix} \neq 0 \qquad (4.1)$$

for all distinct $\tau_0, ..., \tau_r \in J$, and all positive integers $\mu_0, ..., \mu_r$ whose sum is equal to $n+1$. Equivalently, each function $E \in \mathscr{E}$ is completely determined as soon as the quantities $E^{(j)}(\tau_i)$, $i=0, ..., r$, $j=0, ..., \mu_i-1$, are known, where the τ_i's and the μ_i's are chosen as above.

Observe that any EC space on J is a W-space on J. A classical result ([6]) states that $\mathscr{E}$ is an $(n+1)$-dimensional EC space on J iff there exist $n+1$ positive functions $w_0, ..., w_n \in C^\infty(J)$ such that

$$\mathscr{E} := \operatorname{Ker} DL_n, \qquad (4.2)$$

D standing for the ordinary differentiation operator, and the differential operators $L_0, ..., L_n$ being defined recursively on $C^\infty(J)$ by

$$L_0 E := \frac{1}{w_0} E, \qquad L_i E := \frac{1}{w_i} (L_{i-1} E)', \qquad i=1, ..., n. \qquad (4.3)$$

Then, $(w_0, ..., w_n)$ is said to be a system of *weight functions* associated with $\mathscr{E}$, which we shall write $\mathscr{E} = EC(w_0, ..., w_n)$. For instance, the space $\mathscr{P}_n$ of polynomials of degree less than or equal to n is the EC space on $\mathbb{R}$ associated with constant weight functions. Let us observe that different systems of weight functions may lead to the same EC space.

The dual space of an EC space is also an EC space (see [15]). More precisely, if $\mathscr{E} = EC(w_0, ..., w_n)$, its dual space $\mathscr{E}^*$ is the EC space associated with the weight functions $\hat{w}_0, ..., \hat{w}_n$ given by

$$\hat{w}_0 := \frac{1}{w_0 \cdots w_n}, \qquad \hat{w}_i := w_{n+1-i}, \qquad i = 1, ..., n. \tag{4.4}$$

Let us consider the operators $\hat{L}_j$, $j = 0, ..., n$, which are defined from $\hat{w}_0, ..., \hat{w}_n$ similarly to (4.3). Then, any basis $(E_0, ..., E_n)$ of $\mathscr{E}$ and its dual system $(E_0^*, ..., E_n^*)$, satisfy

$$\mathscr{L}_{\vec{E}}(t)^T \cdot \hat{\mathscr{L}}_{\vec{E}^*}(t) = \mathscr{R} \qquad \text{for all} \quad t \in J, \tag{4.5}$$

where, for $t \in J$, $\mathscr{L}_{\vec{E}}(t)$ and $\hat{\mathscr{L}}_{\vec{E}^*}(t)$ denote the square matrices of order $n+1$ defined by

$$(\mathscr{L}_{\vec{E}}(t))_{i,j} := L_j E_i(t), \qquad (\hat{\mathscr{L}}_{\vec{E}^*}(t))_{i,j} := \hat{L}_j E_i^*(t), \qquad i, j = 0, ..., n, \tag{4.6}$$

In relation (4.5), $\mathscr{R}$ stands for the antidiagonal matrix $(1, ..., (-1)^n)$ introduced in (2.25). For the proof of this result, we refer to [15].

4.2. *The Canonical Bilinear Form Associated with a Piecewise Smooth EC Space*

In the following, $\mathscr{U}$ will denote a piecewise smooth EC space of dimension $n+1$. Again, the connections are supposed to be expressed by (2.5), with connection matrices assumed to belong to $\mathscr{M}$. For each $i = 0, ..., q$, we can select a system of positive weight functions $w_0^i, ..., w_n^i \in C^\infty(I_i)$ so that $\mathscr{U}_i = EC(w_0^i, ..., w_n^i)$. Moreover, without any loss of generality, we can assume that

$$w_j^{l-1}(t_l) = w_j^l(t_l), \qquad j = 0, ..., n, \quad l = 1, ..., q. \tag{4.7}$$

We shall denote by L_j^i, $j = 0, ..., n$, the differential operators on $C^\infty(I_i)$ defined from the weight functions $w_0^i, ..., w_n^i$ by means of formulae similar to (4.3). Let us now introduce the following notations. Given $U \in \mathscr{U}$, $x \in I$, and ε, such that $x^\varepsilon \in I_i$,

$$\Lambda_n U(x^\varepsilon) := (L_0^i U(x), L_1^i U(x^\varepsilon), ..., L_n^i U(x^\varepsilon))^T, \tag{4.8}$$

the first term of the right hand side of (4.8) being well-defined because of (4.7). Moreover, if $(U_0, ..., U_n)$ is a basis of $\mathscr{U}$, extending (4.6), we shall set, for $x^\varepsilon \in I_i$,

$$\mathscr{L}_{\vec{U}}(x^\varepsilon) := \begin{pmatrix} L_0^i U_0(x) & L_1^i U_0(x^\varepsilon) & \cdots & L_n^i U_0(x^\varepsilon) \\ L_0^i U_1(x) & L_1^i(U_1(x^\varepsilon) & \cdots & L_n^i U_1(x^\varepsilon) \\ \vdots & \vdots & \ddots & \vdots \\ L_0^i U_n(x) & L_1^i U_n(x^\varepsilon) & \cdots & L_n^i U_n(x^\varepsilon) \end{pmatrix}. \tag{4.9}$$

Let us now consider the weight functions $(\hat{w}_0^i, ..., \hat{w}_n^i)$ defined on I_i by

$$\hat{w}_0^i := \frac{1}{w_0^i \cdots w_n^i}, \qquad \hat{w}_j^i := w_{n+1-j}^i, \quad j = 1, ..., n. \tag{4.10}$$

As in (4.3), with $(\hat{w}_0^i, ..., \hat{w}_n^i)$ we can associate differential operators on $C^\infty(I_i)$ that we shall denote by $\hat{L}_j^i$, $j = 0, ..., n$. From these operators, given $x^\varepsilon \in I$, it will be possible to define both $\hat{\Lambda}_n U^*(x^\varepsilon)$ for any $U^* \in \mathscr{U}^*$ similarly to (4.8), and $\hat{\mathscr{L}}_{\vec{U}^*}(x^\varepsilon)$ for any basis $(U_0^*, ..., U_n^*)$ of $\mathscr{U}^*$ similarly to (4.9). Then, the results recalled in the previous subsection lead to the following statement.

Proposition 4.1. *If $\mathscr{U}$ is a piecewise smooth EC space such that $\mathscr{U}_i = EC(w_0^i, ..., w_n^i)$, then its dual space $\mathscr{U}^*$ is a piecewise smooth EC space such that $\mathscr{U}_i^* = EC(\hat{w}_0^i, ..., \hat{w}_n^i)$. Moreover, any basis $(U_0, ..., U_n)$ of $\mathscr{U}$ and its dual system $(U_0^*, ..., U_n^*)$ satisfy*

$$\mathscr{L}_{\vec{U}}(x^\varepsilon)^T \cdot \hat{\mathscr{L}}_{\vec{U}^*}(x^\varepsilon) = \mathscr{R} \qquad \textit{for all} \quad x^\varepsilon \in I. \tag{4.11}$$

On the other hand, according to the definition of the operators L_j^i, we can write

$$\Lambda_n U(x^\varepsilon) = \mathscr{C}_n(x^\varepsilon) \cdot \Delta_n U(x^\varepsilon), \tag{4.12}$$

where, for $x^\varepsilon \in I_i$, $\mathscr{C}_n(x^\varepsilon)$ is a lower triangular matrix the diagonal of which is

$$\left(\frac{1}{w_0^i(x) \cdots w_j^i(x)}\right)_{j=0, ..., n}. \tag{4.13}$$

Now, instead of using the ordinary derivatives, the connections in the space $\mathscr{U}$ can be expressed by means of the differential operators L_j^i. More precisely, the space $\mathscr{U}$ can be described as the space of all functions

$U: I \to \mathbb{R}$ the restriction of which to I_i belongs to $\mathscr{U}_i = EC(w_0^i, ..., w_n^i)$, $i = 0, ..., q$, and which satisfy

$$\Lambda_n U(t_l^+) = N_l \cdot \Lambda_n U(t_l^-), \tag{4.14}$$

where

$$N_l := \mathscr{C}_n(t_l^+) \cdot M_l \cdot \mathscr{C}_n(t_l^-)^{-1}. \tag{4.15}$$

On account of (4.15), (4.13), and (4.7), matrix N_l is regular and has the same first row and last column as M_l. Proposition 4.1 leads to the following result.

Corollary 4.2. *If the connections in $\mathscr{U}$ are given by* (4.14), *where $N_1, ..., N_q \in \mathscr{M}$, the connections in the dual space $\mathscr{U}^*$ are given by*

$$\hat{\Lambda}_n U^*(t_l^+) = N_l^* \cdot \hat{\Lambda}_n U^*(t_l^-), \tag{4.16}$$

where matrices $N_1^*, ..., N_q^* \in \mathscr{M}$ and are defined by

$$N_l^* := \mathscr{R}^T \cdot N_l^{-T} \cdot \mathscr{R}. \tag{4.17}$$

Using Theorem 2.6, we can also obtain a new expression of the canonical bilinear form.

Corollary 4.3. *For any $U \in \mathscr{U}$ and any $U^* \in \mathscr{U}^*$, we have*

$$[U, U^*] = \Lambda_n U(a^\varepsilon)^T \cdot \mathscr{R} \cdot \Lambda_n U^*(a^\varepsilon), \tag{4.18}$$

for any $a^\varepsilon \in I$.

Proof. The proof is similar to that of Proposition 2.10. Given a basis $(U_0, ..., U_n)$ of $\mathscr{U}$ and its dual system $(U_0^*, ..., U_n^*)$, from (2.16) it can be proved that, for any $U \in \mathscr{U}$ and any $U^* \in \mathscr{U}^*$,

$$[U, U^*] = \Lambda_n U(a^\varepsilon)^T \cdot \mathscr{L}_{\vec{U}}(a^\varepsilon)^{-1} \cdot \hat{\mathscr{L}}_{\vec{U}^*}(a^\varepsilon)^{-T} \cdot \hat{\Lambda}_n U^*(a^\varepsilon). \tag{4.19}$$

Let us choose the basis $(U_0, ..., U_n)$ of $\mathscr{U}$ characterized by $\mathscr{L}_{\vec{U}}(a^\varepsilon) = \mathscr{I}_{n+1}$. By (4.11), its dual system satisfies $\hat{\mathscr{L}}_{\vec{U}^*}(a^\varepsilon) = \mathscr{R}$. Therefore, in that case, (4.19) reduces to (4.18) on account of the equality $\mathscr{R}^{-T} = \mathscr{R}$. ∎

Remarks 4.4. (i) Let us observe that (4.18) can also be written

$$[U, U^*] = \sum_{k=0}^{n} L_k^i U(a^\varepsilon)(-1)^{n-k} \hat{L}_{n-k}^i U^*(a^\varepsilon) \tag{4.20}$$

for all $a \in I$ such that $a^{\varepsilon} \in I_i$. In the piecewise polynomial case, i.e., when all the weight functions associated with each section are constant, this formula is nothing but (2.30).

(ii) In particular, the previous result implies that the right hand side of (4.20) is independent not only of the chosen subinterval I_i and of the chosen point a provided that $a^{\varepsilon} \in I_i$, but also of the weight functions $(w_0^i, ..., w_n^i)$ defining $\mathscr{U}_i$ which are involved in (4.20) through the L_j^i's and the $\hat{L}_j^i$'s.

5. APPLICATION TO BLOSSOMING

In this section we shall use a fundamental result due to P. J. Barry [1]: in case the connection matrices (with respect to the differential operators L_j^i) are lower triangular and *totally positive* (i.e., all their minors are nonnegative), then the number of zeros of any nonzero element of an $(n+1)$-dimensional piecewise smooth EC space is bounded by n.

5.1. *Blossoming*

As in the previous section, $\mathscr{U}$ will denote the piecewise EC space such that $\mathscr{U}_i := EC(w_0^i, ..., w_n^i)$, $i = 0, ..., q$, in which the connections are expressed either by (2.5) or (4.14). Again, the weight functions are assumed to satisfy (4.7). From now on, we suppose that:

- for $i = 0, ..., q$, $w_0^i \equiv 1$ on I_i (which implies, by applying (4.2) that $\mathscr{U}_i$ contains the functions constant on I_i),
- for $l = 1, ..., q$, the connection matrix N_l is regular and lower triangular, its first and last diagonal elements are equal to 1 and its first column is equal to $(1, 0, ..., 0)^T$ (this latter assumption implying that $\mathscr{U}$ also contains the constant functions).

Under these assumptions, by (4.15), the corresponding matrices M_l are also lower triangular. Therefore, if a function $U \in \mathscr{U}$ vanishes at t_l^+ with multiplicity $r \leqslant n$, it also vanishes at t_l^- with the same multiplicity and conversely. If so, t_l will simply be said to be a zero of order r of U, and U to vanish at t_l^r. Moreover, since $\mathscr{U}$ is a piecewise smooth W-space on I any zero of U on I is necessarily of order less than or equal to n, unless U is the zero function. Let us denote by $Z(\mathscr{U})$ the upper bound (possibly infinite) of the number of zeros (counted with multiplicities) of the nonzero elements of the space $\mathscr{U}$.

The assumptions on N_l imply that $N_l^* = \mathscr{R}^T \cdot N_l^{-T} \cdot \mathscr{R}$ is also lower triangular so that the number $Z(\mathscr{U}^*)$ is also well-defined. Moreover, observe that the last row of N_l^* is equal to $(0, ..., 0, 1)$.

Let us choose once and for all a basis $(U_0, ..., U_n)$ in $\mathscr{U}$, with $U_0 = \mathbb{1}$, and denote its dual system by $(U_0^*, ..., U_n^*)$. Consider the n-dimensional space $\mathscr{V}^* := \text{span}(U_1^*, ..., U_n^*)$. Since $w_0^i \equiv 1$ on I_i, one can verify that the restriction $\mathscr{V}_i^*$ of the space $\mathscr{V}^*$ to I_i is the dual space of $D\mathscr{U}_i$. Now, since $\mathscr{U}_i := EC(w_0^i, ..., w_n^i)$, by using the results recalled in Subsection 4.1, it can be checked that $D\mathscr{U}_i = EC(w_1^i, ..., w_n^i)$ and $\mathscr{V}_i^* = EC(\hat{w}_0^i, ..., \hat{w}_{n-1}^i)$, where the weight functions $\hat{w}_0^i, ..., \hat{w}_{n-1}^i$ are those defined in (4.10). On account of the last rows of matrices N_l^*, we can conclude that $\mathscr{V}^*$ is an n-dimensional piecewise smooth EC space on I, in which the connection conditions are given by

$$\hat{\Lambda}_{n-1} V^*(t_l^+) = Q_l \cdot \hat{\Lambda}_{n-1} V^*(t_l^-), \qquad l = 1, ..., q, \tag{5.1}$$

where, for $l = 1, ..., q$, Q_l is the $n \times n$ lower triangular regular matrix obtained by suppressing the last row and the last column of N_l^*. Therefore, the number $Z(\mathscr{V}^*)$ is also well-defined.

LEMMA 5.1. *In addition to the assumptions developed above, suppose that each N_l, $l = 1, ..., q$ is totally positive. Then we have $Z(\mathscr{U}) \leqslant n$, $Z(\mathscr{U}^*) \leqslant n$ and $Z(\mathscr{V}^*) \leqslant n-1$.*

Proof. The total positivity of N_l implies that of N_l^* [1, Theorem 5], hence also that of Q_l. Therefore, Theorem 8 of [1] gives the desired result, directly for the two spaces $\mathscr{U}$ and $\mathscr{U}^*$, and after slightly adapting its assumptions for the space $\mathscr{V}^*$. ∎

Throughout this section we additionally suppose that:

- for $l = 1, ..., q$, the connection matrix N_l is totally positive.

In fact, Lemma 5.1 means that under this total positivity assumption on the connection matrices, the $(n+1)$-dimensional piecewise EC space $\mathscr{U}$ behaves like an $(n+1)$-dimensional EC space. It is actually the key-point which will enable us to develop the blossoming principle in piecewise EC spaces by extending all the results obtained in [19] for a single EC space.

Condition $Z(\mathscr{U}) \leqslant n$ can be equivalently interpreted as follows: a function $U \in \mathscr{U}$ is uniquely determined by $n+1$ data $U(\tau_i)$, $U'(\tau_i^{\varepsilon_i})$, ..., $U^{(\mu_i - 1)}(\tau_i^{\varepsilon_i})$, $i = 0, ..., r$, for given distinct points $\tau_0, ..., \tau_r \in I$, given positive integers $\mu_0, ..., \mu_r$ whose sum is equal to $n+1$ and given ε_i's such that $\tau_i^{\varepsilon_i} \in I$. A similar property can be stated for $\mathscr{U}^*$ and $\mathscr{V}^*$.

If an n-tuple $\mathscr{T}$ satisfies $\mathscr{T}^{ord} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$, a function $\Psi^* \in \mathscr{U}^*$ will be said *to vanish* (*with multiplicities*) *on* $\mathscr{T}$ if $\Psi^{*(k)}(\tau_i) = 0$ for $k = 0, ..., \mu_i - 1$, $i = 1, ..., r$. Although expressed in a different way, the following result is an implicit consequence of [1, Th. 9].

LEMMA 5.2. *Given an n-tuple* $\mathcal{T} \in I^n$, *there exists a unique element* $\Psi^*_{\mathcal{T}} \in \mathcal{U}^*$ *such that*

$$\Psi^*_{\mathcal{T}} \text{ vanishes on } \mathcal{T}, \qquad [\mathbb{1}, \Psi^*_{\mathcal{T}}] = 1. \tag{5.2}$$

Proof. For any $\Psi^* = \sum_{i=0}^{n} \alpha_i^* U_i^* \in \mathcal{U}^*$, $[\mathbb{1}, \Psi^*] = [U_0, \Psi^*] = \alpha_0^*$. Then, Ψ^* satisfies the normalization condition $[\mathbb{1}, \Psi^*] = 1$ (i.e., $\alpha_0^* = 1$), iff the function $\Phi^* := \Psi^* - U_0^*$ belongs to the n-dimensional space $\mathcal{V}^*$. So, if $\mathcal{T}^{ord} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$, it only remains to prove that, for a given choice of ε_i's, there exists exactly one element $\Phi^*_{\mathcal{T}} \in \mathcal{V}^*$ such that

$$\Phi^{*(j)}_{\mathcal{T}}(\tau_i^{\varepsilon_i}) = -U_0^{*(j)}(\tau_i^{\varepsilon_i}), \qquad 1 \leqslant i \leqslant r, \quad 0 \leqslant j \leqslant \mu_i - 1.$$

As noticed above, this is a direct consequence of Lemma 5.1. ∎

DEFINITION AND THEOREM 5.3. *Let F be an element of* $\mathcal{U}$. *By setting, for all n-tuple* $\mathcal{T} \in I^n$,

$$f(\mathcal{T}) := [F, \Psi^*_{\mathcal{T}}], \tag{5.3}$$

where $\Psi^*_{\mathcal{T}}$ *is the only element of* $\mathcal{U}^*$ *satisfying* (5.2), *we define a function* $f\colon I^n \to \mathbb{R}$ *which will be called* the blossom *of F. It is a symmetric function such that*

$$f(x^n) = F(x) \qquad \text{for all} \quad x \in I. \tag{5.4}$$

Proof. The symmetry of f is clear. On the other hand, since E is the reproducing function associated with $\mathcal{U}$, function $E(x, \cdot)$ belongs to $\mathcal{U}^*$ and vanishes on (x^n). From $[\mathbb{1}, E(x, \cdot)] = \mathbb{1}(x) = 1$, we can thus conclude that

$$\Psi^*_{(x^n)} = E(x, \cdot). \tag{5.5}$$

So, by (5.3), $f(x^n) := [F, E(x, \cdot)]$. The reproducing property of E leads to (5.4). ∎

Remarks 5.4. (i) According to (4.20), the normalization condition $[\mathbb{1}, \Psi^*_{\mathcal{T}}] = 1$ can be written

$$\hat{L}_n^i \Psi^*_{\mathcal{T}}(a^\varepsilon) = (-1)^n, \tag{5.6}$$

for any $i \in \{0, ..., q\}$ and any a such that $a^\varepsilon \in I_i$. For instance, in the piecewise polynomial case (5.6) yields $\Psi^{*(n)}_{\mathcal{T}}(a^\varepsilon) \equiv (-1)^n$.

(ii) From (4.20) it can also be deduced that the blossom f of $F \in \mathscr{U}$ can also be written as

$$f(\mathscr{T}) = \sum_{k=0}^{n} L_k^i U(a^\varepsilon)(-1)^{n-k} \hat{L}_{n-k}^i \Psi_{\mathscr{T}}^*(a^\varepsilon)$$

$$\text{for all} \quad i = 0, ..., s \text{ and all } a^\varepsilon \in I_i. \tag{5.7}$$

This is the expression used by P. J. Barry [1] in order to introduce a notion of blossom. Contrary to what such a formula might seem to induce, according to Definition 5.3, the blossom does not depend on the particular systems of weight functions defining the EC spaces $\mathscr{U}_i$.

(iii) As a first obvious example, observe that the blossom of function $\mathbb{1}$ is equal to 1 everywhere on I^n: Indeed, this is a clear consequence of the normalization condition $[\mathbb{1}, \Psi_{\mathscr{T}}^*] = 1$.

(iv) For all $y \in I$, the blossom $e(\cdot\,; y): I^n \to \mathbb{R}$ of $E(\cdot, y)$ being defined by $e(\mathscr{T}; y) := [E(\cdot, y), \Psi_{\mathscr{T}}^*]$, the reproducing property of E leads to

$$e(\mathscr{T}; y) = \Psi_{\mathscr{T}}^*(y). \tag{5.8}$$

5.2. *Pseudo-affinity of the Partial Blossoms*

In this subsection, we shall prove that the blossom f of any $F \in \mathscr{U}$ is a pseudo-affine function with respect to each variable, in the sense indicated by the following theorem.

THEOREM 5.5. *Given an $(n-1)$-tuple $\mathscr{T} \in I^{n-1}$ and two distinct points a, b in I, there exists a function β (depending on $\mathscr{T}$, a, b), piecewise smooth and strictly monotone on I, with $\beta(a) = 0$, $\beta(b) = 1$, such that the blossom f of any $F \in \mathscr{U}$ satisfies, for all $x \in I$,*

$$f(\mathscr{T}, x) = (1 - \beta(x))\, f(\mathscr{T}, a) + \beta(x)\, f(\mathscr{T}, b). \tag{5.9}$$

Proof. Since, for all $x \in I$, $f(\mathscr{T}, x) = [F, \Psi_{(\mathscr{T}, x)}^*]$, (5.9) will be a straightforward consequence of the following three lemmas, the first two of them having already been stated by P. J. Barry in a different form [1]. ∎

LEMMA 5.6. *Given an $(n-1)$-tuple $\mathscr{T} \in I^{n-1}$, with $\mathscr{T}^{ord} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$, and two distinct points a, b in I, the function $\Psi^* := \Psi_{(\mathscr{T}, a)}^* - \Psi_{(\mathscr{T}, b)}^*$ vanishes exactly on $\mathscr{T}$, in the sense that Ψ^* both vanishes (with multiplicities) on $\mathscr{T}$, and satisfies*

$$\Psi^*(x) \neq 0 \quad \text{if} \quad x \notin \{\tau_1, ..., \tau_r\}, \qquad \Psi^{*(\mu_i)}(\tau_i^{\varepsilon_i}) \neq 0, \qquad i = 1, ..., r. \tag{5.10}$$

Proof. By Lemma 5.1, $Z(\mathscr{U}^*) \leqslant n$. Consequently, since $a \neq b$, on account of their zeros, the two functions $\Psi^*_{(\mathscr{T},a)}$ and $\Psi^*_{(\mathscr{T},b)}$ are linearly independent. Therefore, in particular $\Psi^* \not\equiv 0$. Due to the two normalization conditions $[\mathbb{1}, \Psi^*_{(\mathscr{T},a)}] = 1$ and $[\mathbb{1}, \Psi^*_{(\mathscr{T},b)}] = 1$, Ψ^* is thus a nonzero element of the space $\mathscr{V}^*$. Consequently, according to Lemma 5.1, it has at most $(n-1)$ zeros in I. Now, $\Psi^*_{(\mathscr{T},a)}$ and $\Psi^*_{(\mathscr{T},b)}$ both vanishing (with multiplicities) on the $(n-1)$-tuple $\mathscr{T}$, so does Ψ^*. Consequently, it cannot vanish elsewhere (with multiplicities), in the sense of (5.10). ∎

LEMMA 5.7. *With the same assumptions as in Lemma 5.6, for all $x \in I$, we have*

$$\Psi^*_{(\mathscr{T},x)} = \alpha(x)\, \Psi^*_{(\mathscr{T},\alpha)} + \beta(x)\, \Psi^*_{(\mathscr{T},b)}, \tag{5.11}$$

where, if $x \notin \{\tau_1, ..., \tau_r\}$,

$$\alpha(x) := \frac{\Psi^*_{(\mathscr{T},b)}(x)}{\Psi^*_{(\mathscr{T},b)}(x) - \Psi^*_{(\mathscr{T},a)}(x)}, \qquad \beta(x) := \frac{\Psi^*_{(\mathscr{T},a)}(x)}{\Psi^*_{(\mathscr{T},a)}(x) - \Psi^*_{(\mathscr{T},b)}(x)}, \tag{5.12}$$

and, for $i = 1, ..., r$,

$$\alpha(\tau_i) := \frac{\Psi^{*(\mu_i)}_{(\mathscr{T},b)}(\tau_i^{\varepsilon_i})}{\Psi^{*(\mu_i)}_{(\mathscr{T},b)}(\tau_i^{\varepsilon_i}) - \Psi^{*(\mu_i)}_{(\mathscr{T},a)}(\tau_i^{\varepsilon_i})}, \qquad \beta(\tau_i) := \frac{\Psi^{*(\mu_i)}_{(\mathscr{T},a)}(\tau_i^{\varepsilon_i})}{\Psi^{*(\mu_i)}_{(\mathscr{T},a)}(\tau_i^{\varepsilon_i}) - \Psi^{*(\mu_i)}_{(\mathscr{T},b)}(\tau_i^{\varepsilon_i})}. \tag{5.13}$$

Proof. Since $Z(\mathscr{U}^*) \leqslant n$, the space of all functions in $\mathscr{U}^*$ vanishing on $\mathscr{T}$ is 2-dimensional. Therefore, there exist two unique real numbers $\alpha(x)$, $\beta(x)$ such that

$$\Psi^*_{(\mathscr{T},x)} = \alpha(x)\, \Psi^*_{(\mathscr{T},a)} + \beta(x)\, \Psi^*_{(\mathscr{T},b)}. \tag{5.14}$$

In order to determine these two numbers, observe that (5.14) easily yields a first linear relation between them by means of the normalization conditions on the three functions $\Psi^*_{(\mathscr{T},x)}$, $\Psi^*_{(\mathscr{T},a)}$, and $\Psi^*_{(\mathscr{T},b)}$:

$$\alpha(x) + \beta(x) = 1. \tag{5.15}$$

With an argument similar to that used in [19, Lemma 4.10] (to which we refer for more details), a second one can be obtained from the fact that $\Psi^*_{(\mathscr{T},x)}$ vanishes on $(\mathscr{T}, x)$. This leads either to

$$\alpha(x)\, \Psi^*_{(\mathscr{T},a)}(x) + \beta(x)\, \Psi^*_{(\mathscr{T},b)}(x) = 0, \tag{5.16}$$

or to

$$\alpha(x)\, \Psi^{*(\mu_i)}_{(\mathscr{T},a)}(\tau_i^{\varepsilon_i}) + \beta(x)\, \Psi^{*(\mu_i)}_{(\mathscr{T},b)}(\tau_i^{\varepsilon_i}) = 0, \tag{5.17}$$

depending on whether $x \notin \{\tau_1, ..., \tau_r\}$ or $x = \tau_i$, $1 \leqslant i \leqslant r$. In either case, due to Lemma 5.6, the determinant of the so obtained system is not equal to 0, and solving it gives (5.12) and (5.13). ∎

LEMMA 5.8. *Functions α and β defined by* (5.12) *and* (5.13) *are piecewise smooth and strictly monotone on I.*

Proof. The unicity of the couple $(\alpha(x), \beta(x))$ satisfying (5.14) for a given $x \in I$ proves that the two functions α and β are well-defined on I (i.e., they do not depend on possible ε's). As to why these two functions are C^∞ on each subinterval I_i, we refer to [16].

Now, if x_1 and x_2 are two distinct points in I, $\beta(x_1) \neq \beta(x_2)$. Otherwise, we would have $\Psi^*_{(\mathcal{T}, x_1)} = \Psi^*_{(\mathcal{T}, x_2)}$ and this nonzero element of the space $\mathcal{U}^*$ would have $n+1$ zeros in I, which is contrary to what Lemma 5.1 says. Hence, β being one-to-one and C^0 on I, it is a strictly monotone function. ∎

5.3. *Chebyshev–Bézier Points and Chebyshev–Bernstein Basis*

PROPOSITION 5.9. *Given $n+1$ n-tuples $\mathcal{T}_0, ..., \mathcal{T}_n \in I^n$ and $n+1$ functions $D_0, ..., D_n \in \mathcal{U}$, the following four statements are equivalent*:

(i) $E(x, y) = \sum_{i=0}^n D_i(x)\, \Psi^*_{\mathcal{T}_i}(y)$ *for all* $x, y \in I$,

(ii) *the blossoms $d_0, ..., d_n$ of functions $D_0, ..., D_n$ satisfy*

$$d_i(\mathcal{T}_j) = \delta_{ij}, \qquad i, j = 0, ..., n, \tag{5.18}$$

(iii) $(\Psi^*_{\mathcal{T}_0}, ..., \Psi^*_{\mathcal{T}_n})$ *is a basis of $\mathcal{U}^*$ and $(-1)^n (D_0, ..., D_n)$ is its dual system,*

(iv) *for all $F \in \mathcal{U}$,*

$$F = \sum_{i=0}^n f(\mathcal{T}_i)\, D_i. \tag{5.19}$$

Moreover, as soon as these properties hold, we have

$$\sum_{i=0}^n D_i = \mathbb{1}. \tag{5.20}$$

Proof. The equivalence between the first three properties is just a consequence of Proposition 2.9 because

$$d_i(\mathcal{T}_j) := [D_i, \Psi^*_{\mathcal{T}_j}], \qquad i, j = 0, ..., n.$$

Suppose that (i) holds. Then, since E is the reproducing function associated with $\mathscr{U}$, we have, for all $F \in \mathscr{U}$ and all $x \in I$,

$$F(x) = [F, E(x, \cdot)] = \sum_{i=0}^{n} D_i(x)[F, \Psi^*_{\mathscr{T}_i}],$$

which is exactly (5.19).

Conversely, when applying (5.19) to $E(\cdot, y)$, $y \in I$, we obtain

$$E(\cdot, y) = \sum_{i=0}^{n} e(\mathscr{T}_i; y)\, D_i. \tag{5.21}$$

According to (5.8), (5.21) is nothing but condition (i).

Finally, as soon as (5.19) is valid, (5.20) is obtained by applying it to function $\mathbb{1}$. ∎

PROPOSITION 5.10. *Let* $a_{-n+1} \leqslant a_{-n+2} \leqslant \cdots \leqslant a_0 < a_1 \leqslant \cdots \leqslant a_n$ *be* $2n$ *points of* I *and consider the following* n*-tuples*

$$\mathscr{T}_i := (a_{i-n+1}, ..., a_i), \qquad i = 0, ..., n. \tag{5.22}$$

Then, the $n+1$ *functions* $\Psi^*_{\mathscr{T}_i}$, $i = 0, ..., n$, *form a basis of* $\mathscr{U}^*$ *and the corresponding basis* $(D_0, ..., D_n)$ (*defined, for instance, by* (i) *of Proposition* 5.9) *satisfies* $D_i(x) > 0$ *for all* $x \in\,]a_0, a_1[$.

The proof of this result will be a consequence of the following lemma which is an extension of the classical de Boor algorithm.

LEMMA 5.11. *With the same assumptions as in Proposition* 5.10, *given* $x \in I$, *the value at* x *of any* $F \in \mathscr{U}$ *can be obtained as an affine combination of the points* $f(\mathscr{T}_0), ..., f(\mathscr{T}_n)$, *the coefficients of which do not depend on* F. *Moreover, these coefficients are strictly positive as soon as* $x \in\,]a_0, a_1[$.

Proof. For given integers k and i, $0 \leqslant k \leqslant n-1$, $0 \leqslant i \leqslant n-k-1$, we have $a_{i-n+1+k} \leqslant a_0 < a_1 \leqslant a_{i+1}$. Therefore, we can apply Theorem 5.5 with the $(n-1)$-tuple $\mathscr{T} = (a_{i-n+2+k}, ..., a_i, x^k)$ as

$$f(\mathscr{T}, x) = (1 - \beta^k_i(x))\; f(\mathscr{T}, a_{i-n+1+k}) + \beta^k_i(x)\; f(\mathscr{T}, a_{i+1}), \qquad x \in I, \tag{5.23}$$

where function β^k_i does not depend on $F \in \mathscr{U}$ and $\beta^k_i(x) \in\,]0, 1[$ for $x \in\,]a_{i-n+1+k}, a_{i+1}[$.

For a given $x \in I$, let us introduce the real numbers

$$P^k_i(x) := f(a_{i-n+1+k}, ..., a_i, x^k), \qquad i = 0, ..., n-k. \tag{5.24}$$

According to (5.23), these points $P_i^k(x)$ satisfy the recursive relation

$$P_i^{k+1}(x) = (1 - \beta_i^k(x))\, P_i^k(x) + \beta_i^k(x)\, P_{i+1}^k(x),$$
$$k = 0, ..., n-1, \quad i = 0, ..., n-k-1. \tag{5.25}$$

Furthermore, we have in particular

$$P_i^0(x) = f(\mathscr{T}_i), \quad i = 0, ..., n, \qquad P_0^k(x) = f(x^n) = F(x). \tag{5.26}$$

Relations (5.25) describe what we shall refer to as *the Chebyshev–de Boor algorithm* (related to the $2n$ chosen points). This algorithm allows us to compute (in n steps) the value of a function $F \in \mathscr{U}$ at a point $x \in I$ as an affine combination of the points $f(\mathscr{T}_0), ..., f(\mathscr{T}_n)$, this affine combination being in fact a strictly convex one as soon as $x \in]a_0, a_1[$ since $]a_0, a_1[\subset]a_{i-n+1+k}, a_{i+1}[$ for all $k = 0, ..., n-1$, $i = 0, ..., n-k-1$. Actually, the coefficients of these combinations do not depend on the function F since this holds for each β_i^k. ∎

Proof of Proposition 5.10. As a consequence of the previous lemma, the linear map $\Phi: \mathscr{U} \to \mathbb{R}^{n+1}$ defined by

$$\Phi(F) = (f(\mathscr{T}_i))_{i=0, ..., n} \tag{5.27}$$

is one-to-one on $\mathscr{U}$. Using the definition of the blossom, this means that the following $n+1$ linear forms on $\mathscr{U}$

$$[\cdot, \Psi^*_{\mathscr{T}_i}], \qquad i = 0, ..., n, \tag{5.28}$$

are linearly independent, which implies the linear independence of $\Psi^*_{\mathscr{T}_0}, ..., \Psi^*_{\mathscr{T}_n}$.

In application of the Chebyshev–de Boor algorithm, for a given i, and a given $x \in]a_0, a_1[$, $D_i(x)$ is a strictly convex combination of the real numbers $d_i(\mathscr{T}_j)$, $j = 0, ..., n$. Through (5.18), these real numbers are all equal to 0 except that of index i which is equal to 1. Hence, $D_i(x) > 0$. ∎

As a particular case of Proposition 5.10, we can take $a_j = \min(a, b)$ for $j = -n+1, ..., 0$ and $a_j = \max(a, b)$ for $j = 1, ..., n$, where a, b are two distinct points of I. Therefore, the $n+1$ functions $\Psi^*_{(a^{n-i}b^i)}$, $i = 0, ..., n$, form a basis of $\mathscr{U}^*$.

DEFINITION 5.12. Given two distinct points $a, b \in I$, and $F \in \mathscr{U}$, the $n+1$ points

$$P_i := f(a^{n-i}b^i), \qquad i = 0, ..., n, \tag{5.29}$$

will be called *the Chebyshev–Bézier points of F with respect to* (a, b) and the basis $(\mathscr{B}_0, ..., \mathscr{B}_n)$ of $\mathscr{U}$ characterized by

$$E(x, y) = \sum_{i=0}^{n} \mathscr{B}_i(x)\, \Psi^*_{(a^{n-i}b^i)}(y), \qquad x, y \in I, \tag{5.30}$$

will be called *the Chebyshev–Bernstein basis with respect to* (a, b).

In that particular case, the Chebyshev–de Boor algorithm described in the proof of Lemma 5.11 will be called *the Chebyshev–de Casteljau algorithm with respect to* (a, b): it allows the computation in n steps of the values of any $F \in \mathscr{U}$ from its Chebyshev–Bézier points with respect to (a, b).

From Proposition 5.9 it results that the Chebyshev–Bézier basis satisfies $b_i(a^{n-j}b^j) = \delta_{ij}$, $i, j = 0, ..., n$, or equivalently, that

$$F = \sum_{i=0}^{n} f(a^{n-i}b^i)\, \mathscr{B}_i \qquad \text{for all} \quad F \in \mathscr{U}. \tag{5.31}$$

Moreover, Proposition 5.10 proves that, for $i = 0, ..., n$, $\mathscr{B}_i(x) > 0$ as soon as x is strictly located between a and b.

THEOREM 5.13. *The Chebyshev–Bernstein basis with respect to* (a, b) *is characterized by the following two properties*

(i) *for all* $i = 0, ..., n$, $\mathscr{B}_i$ *vanishes on* $(a^i b^{n-i})$,

(ii) $\mathbb{1} = \sum_{i=0}^{n} \mathscr{B}_i$.

Proof. We already know that $\mathbb{1} = \sum_{i=0}^{n} \mathscr{B}_i$. For a given integer i, $0 \leqslant i \leqslant n$, the Chebyshev–Bernstein function $\mathscr{B}_i$ is characterized by the equalities

$$[\mathscr{B}_i, \Psi^*_{(a^{n-j}b^j)}] = \delta_{ij}, \qquad j = 0, ..., n. \tag{5.32}$$

From Proposition 5.10, we know that the $n+1$ functions $\Psi^*_{(a^{n-j}b^j)}$, $j = 0, ..., n$ are linearly independent. Therefore, for a given integer i, functions $\Psi^*_{(a^{n-j}b^j)}$, $j = 0, ..., i-1$ form a basis of the i-dimensional space

$$\mathscr{A}^*_1 := \{ U^* \in \mathscr{U}^* \mid U^* \text{ vanishes on } (a^{n-i+1}) \}.$$

Similarly, functions $\Psi^*_{(a^{n-j}b^j)}$, $j = i+1, ..., n$, form a basis of the $(n-i)$-dimensional space

$$\mathscr{A}^*_2 := \{ U^* \in \mathscr{U}^* \mid U^* \text{ vanishes on } (b^{i+1}) \}.$$

Consequently, it results from (5.32) that $\mathscr{B}_i$ belongs to $\mathscr{A}_1^{*\circ} \cap \mathscr{A}_2^{*\circ}$. This leads to (i) since, by Corollary 2.13,

$$\mathscr{A}_1^{*\circ} := \{ U \in \mathscr{U} \mid U \text{ vanishes on } (a^i) \},$$

$$\mathscr{A}_2^{*\circ} := \{ U \in \mathscr{U} \mid U \text{ vanishes on } (b^{n-i}) \}.$$

Conversely, let $\Psi_0, ..., \Psi_n$ be $n+1$ elements of $\mathscr{U}$ such that Ψ_i vanishes on $(a^i b^{n-i})$, $i = 0, ..., n$. The same argument as previously leads to $[\Psi_i, \Psi^*_{(a^{n-j}b^j)}] = 0$ for $j \neq i$. Consequently, the additional condition $\mathbb{1} = \sum_{i=0}^n \Psi_i$ clearly implies $[\mathbb{1}, \Psi^*_{(a^{n-j}b^j)}] = [\Psi_j, \Psi^*_{(a^{n-j}b^j)}]$. Due to the normalization condition on functions $\Psi^*_{(a^{n-j}b^j)}$, we finally have

$$[\Psi_i, \Psi^*_{(a^{n-j}b^j)}] = \delta_{ij}, \qquad i, j = 0, ..., n,$$

from which we can conclude that $(\Psi_0, ..., \Psi_n)$ is the Chebyshev–Bernstein basis with respect to (a, b). ∎

Remark 5.14. (i) Condition (i) in Theorem 5.13 determines $\mathscr{B}_i$ up to a multiplication by a nonzero real number. Actually, $\mathscr{B}_i$ is the unique element $U \in \mathscr{U}$ which vanishes on $(a^i b^{n-1})$ and satisfies the additional condition $[U, \Psi^*_{(a^{n-i}b^i)}] = 1$. Now, given a function $U \in \mathscr{U}$ vanishing on (a^i) and a function $U^* \in \mathscr{U}^*$ vanishing on (a^{n-i}), we have, by (2.30)

$$[U, U^*] = (-1)^{n-i}\, U^{(i)}(a^\varepsilon)\, U^{*(n-i)}(a^\varepsilon). \tag{5.33}$$

Thus, among all the functions in $\mathscr{U}$ which vanish on $(a^i b^{n-i})$, $\mathscr{B}_i$ is also characterized by

$$\mathscr{B}_i^{(i)}(a^\varepsilon) = \frac{(-1)^{n-i}}{\Psi^{*(n-i)}_{(a^{n-i}b^i)}(a^\varepsilon)}. \tag{5.34}$$

(ii) For instance, $E(\cdot, b)$ and $\mathscr{B}_0$ both vanishing on (b^n), there exists a nonzero real number c such that $\mathscr{B}_0 = cE(\cdot, b)$. Either from (5.34) or more simply by $b_0(a^n) = \mathscr{B}_0(a) = 1$, we can conclude that

$$\mathscr{B}_0 = \frac{E(\cdot, b)}{E(a, b)}. \tag{5.35}$$

5.4. *Blossoming and Contact*

Because of the lower triangular structure of the connection matrices, we can define *the contact of order* $r \leqslant n$ at any point $a \in I$ between two elements $F, G \in \mathscr{U}$ by $F^{(i)}(a^\varepsilon) = G^{(i)}(a^\varepsilon)$, $i = 0, ..., r$, for a given ε such that $a^\varepsilon \in I$: indeed, for $a = t_l$, $1 \leqslant l \leqslant q$, the latter equalities hold for $\varepsilon = -$ iff they hold for $\varepsilon = +$.

As a consequence of Theorem 2.14, we can characterize this contact between two given elements of $\mathscr{U}$ through their blossoms.

THEOREM 5.15. *Let F, G be two elements of $\mathscr{U}$ and f, g their blossoms. Given a point $a \in I$, the following three statements are equivalent:*

(i) *F and G have a contact of order $r \leqslant n$ at a,*

(ii) *F and G have the same $r+1$ first Bézier–Chebyshev points with respect to (a, b), i.e.,*

$$f(a^{n-i}b^i) = g(a^{n-i}b^i), \qquad i = 0, ..., r, \tag{5.36}$$

where b is a given point of I, $b \neq a$,

(iii) *$f(\mathscr{T}) = g(\mathscr{T})$ for all n-tuple $\mathscr{T}$ containing (a^{n-r}).*

Proof. By Theorem 2.14, condition (i) is satisfied iff

$$[F, \Psi^*] = [G, \Psi^*] \qquad \text{for all} \quad \Psi^* \in \mathscr{A}^*, \tag{5.37}$$

where $\mathscr{A}^*$ is the set of all elements of $\mathscr{U}^*$ vanishing on (a^{n-r}). Now, $(\Psi^*_{(a^n)}, ..., \Psi^*_{(a^{n-r}b^r)})$ is a basis of $\mathscr{A}^*$, and $\mathscr{A}^*$ is also spanned by the set of all functions $\Psi^*_{\mathscr{T}}$, where $\mathscr{T}$ contains (a^{n-r}). Therefore, by linearity, condition (5.37) is equivalent either to

$$[F, \Psi^*_{(a^{n-i}b^i)}] = [G, \Psi^*_{(a^{n-i}b^i)}], \qquad i = 0, ..., r,$$

or to

$$[F, \Psi^*_{\mathscr{T}}] = [G, \Psi^*_{\mathscr{T}}] \qquad \text{for all } \mathscr{T} \text{ containing } (a^{n-r}),$$

which corresponds to conditions (ii) and (iii), respectively. ∎

6. EC SPLINES

In this section, we shall obtain a number of results already stated by P. J. Barry [1] (for instance the existence of a B-spline basis). Appearing here as immediate consequences of the general results contained in the previous sections, such proofs are very simple and are thus of interest in their own right.

6.1. *The Blossom of a Spline*

Let us apply the results of the previous sections to the case of EC splines, i.e., W-splines whose sections belong to given EC spaces. So, assume that

• for $i=0, ..., q$, $\mathscr{U}_i = EC(\mathbb{1}, w_1^i, ..., w_n^i)$, where the weight functions $w_1^i, ..., w_n^i \in C^\infty(I_i)$ satisfy the continuous joining condition (4.7),

• for $l=1, ..., q$, B_l is a totally positive regular lower triangular matrix of order n_l+1, $0 \leqslant n_l \leqslant n$, with $(1, 0, ..., 0)^T$ as its first column. In case $n_l = n$ the last diagonal element of B_l is assumed to be equal to 1.

Denote by $\mathscr{S}$ the space of all functions $S: \to \mathbb{R}$ such that $S_{|I_i} \in \mathscr{U}_i$, $i=0, ..., q$ the connections being given by

$$A_{n_l}^l S(t_l^+) = B_l \cdot A_{n_l}^{l-1} S(t_l^-), \qquad l=1, ..., q. \tag{6.1}$$

Following the process developed in Section 3, each B_l will be complemented so as to create a matrix N_l of order $n+1$. Assume that

• For $l=1, ..., q$, N_l is regular, lower triangular, totally positive, its first column is equal to $(1, 0, ..., 0)^T$ and its last diagonal element is equal to 1.

• $\mathscr{U}$ is the piecewise EC space defined from the N_l's as in Section 5, and $\mathscr{U}^*$ its dual space. Again, $\mathscr{U}$ is contained in $\mathscr{S}$.

The space $\mathscr{S}$ being the W-spline space associated with the $\mathscr{U}_i$'s and with the sequence $(A_1, ..., A_q)$ of matrices, where $A_l := \mathscr{C}_{n_l}(t_l^+)^{-1} \cdot B_l \cdot \mathscr{C}_{n_l}(t_l^-)$, we can use the results obtained in Section 3.

PROPOSITION 6.1. *Given a spline function $S \in \mathscr{S}$, for any $i=0, ..., q$, denote by $\tilde{S}_i$ the unique element of $\mathscr{U}$ which satisfies $\tilde{S}_{i\,|\,I_i} = S_{|I_i}$ and by $\tilde{s}_i$ the corresponding blossom. If $\mathscr{T} \in I^n$ is an admissible n-tuple, then all the blossoms $\tilde{s}_l$, $l \in \mathscr{J}(\mathscr{T})$ take the same value on $\mathscr{T}$.*

Proof. As soon as $\mathscr{J}(\mathscr{T})$ contains two consecutive integers $l-1$ and l, $1 \leqslant l \leqslant q$, necessarily t_l appears at least m_l times in $\mathscr{T}$. Consequently, $\Psi^*_{\mathscr{T}}$ vanishes on $(t_l^{m_l})$. Thus, applying Proposition 3.6 proves that all the quantities $[\tilde{S}_l, \Psi^*_{\mathscr{T}}]$, $l \in \mathscr{J}(\mathscr{T})$, are equal, which yields the desired result. ∎

DEFINITION 6.2. With the same notations as in Proposition 6.1, the symmetric function s defined on any admissible n-tuple $\mathscr{T} \in I^n$ by

$$s(\mathscr{T}) := \tilde{s}_l(\mathscr{T}) \qquad \text{for all} \quad l \in \mathscr{J}(\mathscr{T}) \tag{6.2}$$

will be called *the blossom* of S.

THEOREM 6.3. *The blossom s of any spline $S \in \mathscr{S}$ is a symmetric function such that $s(x^n) = S(x)$ for all $x \in I$. For any admissible n-tuple $\mathscr{T}$, we have*

$$s(\mathscr{T}) = \sum_{i=0}^{n} \Lambda_i^l S(a^\varepsilon)(-1)^{n-i} \hat{\Lambda}_{n-i}^l \Psi_{\mathscr{T}}^*(a^\varepsilon) \tag{6.3}$$

for any $l \in \mathscr{J}(\mathscr{T})$, and any $a^\varepsilon \in I_l$. Moreover, if $(x_1, ..., x_{n-1}) \in I^{n-1}$ is an admissible $(n-1)$-tuple, the function $s(x_1, ..., x_{n-1}, \cdot)$ is pseudo-affine in the following sense: for any distinct points $a, b \in \mathscr{D}(x_1, ..., x_{n-1})$, there exists a function β (depending on $x_1, ..., x_{n-1}, a, b$, but not on S) such that, for all $x \in \mathscr{D}(x_1, ..., x_{n-1})$,

$$s(x_1, ..., x_{n-1}, x) = (1 - \beta(x))\, s(x_1, ..., x_{n-1}, a) + \beta(x)\, s(x_1, ..., x_{n-1}, b). \tag{6.4}$$

Proof. Given $x \in I_l$, $S(x) = \tilde{S}_l(x)$. Moreover, the n-tuple (x^n) is admissible and $l \in \mathscr{J}(x^n)$. So, in particular, $s(x^n) = \tilde{s}_l(x^n) = \tilde{S}_l(x) = S(x)$.

Formula (6.3) is thus a straightforward consequence of the definition of $s(\mathscr{T})$ and of (4.20).

Finally, let us choose an integer l in $\mathscr{J}(x_1, ..., x_{n-1})$. Due to the definition of admissibility, for all $x \in \mathscr{D}(x_1, ..., x_{n-1})$, l belongs to $\mathscr{J}(x_1, ..., x_{n-1}, x)$. Consequently, according to Definition 6.2,

$$s(x_1, ..., x_{n-1}, x) = \tilde{s}_l(x_1, ..., x_{n-1}, x).$$

Now, we can apply Theorem 5.5 to $\tilde{s}_l$. ∎

Remark 6.4. This notion of blossom does not depend on the piecewise smooth EC space (with totally positive connections) $\mathscr{U} \subset \mathscr{S}$ (i.e., on the way of complementing the connection matrices B_l). Suppose that $\mathscr{T} \in I^n$ is admissible and that $\mathscr{D}(\mathscr{T}) = [t_i, t_{i+k}]$. In order to calculate $s(\mathscr{T})$ for any $S \in \mathscr{S}$, it is sufficient to know the restriction of $\Psi_{\mathscr{T}}^*$ on $[t_i, t_{i+k}]$. Now, this restriction does not depend on $\mathscr{U}$. As a matter of fact, because of the admissibility of $\mathscr{T}$, for a given integer l, $i < l < i+k$, $\Psi_{\mathscr{T}}^*$ vanishes on $t_l^{m_l}$. Hence, if

$$N_l = \begin{pmatrix} B_l & 0 \\ C_l & D_l \end{pmatrix}, \qquad N_l^* = \begin{pmatrix} B_l^* & 0 \\ C_l^* & D_l^* \end{pmatrix},$$

the two submatrices B_l^* and C_l^* are not involved in the connection condition $\hat{\Lambda}_n \Psi_{\mathscr{T}}^*(t_l^+) = N_l^* \cdot \hat{\Lambda}_n \Psi_{\mathscr{T}}^*(t_l^-)$. Moreover, $D_l^* = \bar{\mathscr{R}}^T \cdot B_l^{-T} \cdot \bar{\mathscr{R}}$, where $\bar{\mathscr{R}}$ is the antidiagonal matrix obtained by suppressing the first m_l rows and last m_l columns of $\mathscr{R}$; hence it depends only on B_l.

6.2. *The Lyche–de Boor Algorithm for Splines and Marsden-Type Identities*

Let us rename the knot vector $T := (t_0^{n+1}, t_1^{m_1}, ..., t_q^{m_q}, t_{q+1}^{n+1})$ as

$$T = (x_{-n}, x_{-n+1}, ..., x_{n+m+1}),$$

so that, in particular $x_{-i} = t_0$ and $x_{m+1+i} = t_{q+1}$ for $i = 0, ..., n$ ($m = \sum_{l=1}^{q} m_l$). When taking any n consecutive points x_i out of the knot vector, we clearly obtain an admissible n-tuple. Let us consider the $n+m+1$ following ones

$$X_j := (x_{j+1}, ..., x_{j+n}), \qquad j = -n, ..., m. \tag{6.5}$$

This leads to the following definition.

Definition 6.5. Given an EC spline $S \in \mathscr{S}$, the $n+m+1$ real numbers

$$Q_j := s(X_j), \qquad j = -n, ..., m, \tag{6.6}$$

will be called *the poles* of S.

It will be possible to reconstruct a spline S from its poles. This requires the following two lemmas.

Lemma 6.6. *For all $j = -n, ..., m$, the domain of the n-tuple X_j is given by*

$$\mathscr{D}(X_j) = [x_j, x_{j+n+1}]. \tag{6.7}$$

Proof. Let $i_0 = 0 < i_1 < \cdots < i_p < i_{p+1} = q+1$ be the sequence of all integers $i \in \{0, ..., q+1\}$ such that $m_i > 0$. Except for $j = -n$ (in which case $X_{-n} = (t_0^n)$) or $j = m$ (in which case $X_m = (t_{n+1}^n)$), each X_j can always be written as

$$X_j = (t_{i_k}^{\alpha} t_{i_{k+1}}^{m_{i_{k+1}}} \cdots t_{i_{k+r}}^{m_{i_{k+r}}} t_{i_{k+r+1}}^{\beta}),$$

with $0 \leqslant \alpha < m_{i_k}$, $0 \leqslant \beta < m_{i_{k+r+1}}$. If so, we have $\mathscr{D}(X_j) = [t_{i_k}, t_{i_{k+r+1}}]$. Moreover, if $\alpha = 0$, the left hand point of X_j is $x_{j+1} = t_{i_{k+1}}$ and we have $x_j = t_{i_k}$. On the contrary, if $\alpha > 0$, $x_{j+1} = t_{i_k}$, but, since $\alpha < m_{i_k}$, we also have $x_j = t_{i_k}$. A similar argument about β eventually yields (6.7).

On the other hand, clearly $\mathscr{D}(X_{-n}) = [t_0, t_{i_1}]$, which can also be written $\mathscr{D}(X_{-n}) = [x_{-n}, x_1]$. Similarly, $\mathscr{D}(X_m) = [t_{i_p}, t_{q+1}] = [x_m, x_{m+n+1}]$. ∎

Lemma 6.7. *Given an integer i, $0 \leqslant i \leqslant q$, there exist exactly $n+1$ integers j such that $i \in \mathscr{J}(X_j)$, namely $j = j_i - n, ..., j_i$, where $j_i := \sum_{l=1}^{i} m_l$ (so that $j_0 = 0$, $j_q = m$).*

Proof. Due to the definition of j_i, x_{j_i} is the greatest knot of nonzero multiplicity to be less than or equal to t_i, whereas $x_{j_{i+1}}$ is the smallest one to be greater or equal to t_{i+1}. In other words, j_i is the unique integer satisfying

$$[t_i, t_{i+1}] \subset [x_{j_i}, x_{j_{i+1}}].$$

Furthermore, $[t_i, t_{i+1}] \subset \mathscr{D}(X_j)$ iff $[x_{j_i}, x_{j_{i+1}}] \subset \mathscr{D}(X_j)$.

In addition, the previous lemma leads to the equivalence

$$[x_{j_i}, x_{j_i+1}] \subset \mathscr{D}(X_j) \Leftrightarrow \begin{cases} x_j \leqslant x_{j_i}, \\ x_{j_i+1} \leqslant x_{j+n+1}. \end{cases}$$

Now, the two conditions $x_j \leqslant x_{j_i}$ and $x_{j_k+1} \leqslant x_{j+n+1}$ clearly hold iff $j \leqslant j_k$ and $j_i + 1 \leqslant j + n + 1$. ∎

Let S be an element of $\mathscr{S}$, and Q_j, $j = -n, ..., m$, its poles. We want to compute $S(x)$ for a given $x \in I$. Actually, if $x \in I_i$, this amounts to computing $\widetilde{S}_i(x)$, where, as previously, $\widetilde{S}_i$ stands for the only element of $\mathscr{U}$ coinciding with S on I_i.

Now, according to both Lemma 6.7 and Definition 6.2, we have

$$Q_j = \tilde{s}_i(X_j), \qquad j = j_i - n, ..., j_i. \tag{6.8}$$

Moreover, the $2n$ consecutive points x_j, $j = j_i - n + 1, ..., j_i + n$, involved in (6.8) satisfy

$$x_{j_i-n+1} \leqslant x_{j_i-n+2} \leqslant \cdots \leqslant x_{j_i} \leqslant t_i < t_{i+1} \leqslant x_{j_i+1} \leqslant x_{j_i+2} \leqslant \cdots \leqslant x_{j_i+n}. \tag{6.9}$$

Therefore, we can apply the Chebyshev–de Boor algorithm to function $\widetilde{S}_i$: thus, $\widetilde{S}_i(x)$ can be computed in n steps as a convex combination of the $n+1$ consecutive poles $Q_{j_i-n}, ..., Q_{j_i}$ and the coefficients of this combination do not depend on S.

Let us mention that the previous algorithm was first obtained by T. Lyche starting from recurrence relations for Chebyshev B-splines [12]. We will therefore refer to it as the *Lyche–de Boor algorithm*. Observe that in case all the multiplicities at the knots are equal to zero, (i.e., when $\mathscr{S} = \mathscr{U}$), it is nothing but the Chebyshev–de Casteljau algorithm with respect to the end points (t_0, t_{q+1}).

As a consequence of the Lyche–de Boor algorithm, the linear map $\Theta: \mathscr{S} \to \mathbb{R}^{n+m+1}$ defined by

$$\Theta(S) = (s(X_j))_{j=-n, ..., m}$$

is one-to-one on $\mathscr{S}$, hence it is an isomorphism since $\mathscr{S}$ is of dimension $n+m+1$. Let $\mathcal{N}_j$ denote the element of $\mathscr{S}$ (the blossom of which will be denoted by n_j) characterized by

$$n_j(X_i)=\delta_{ij}, \qquad i=-n, ..., m. \tag{6.10}$$

Then, functions $\mathcal{N}_j$, $j=-n, ..., m$, form a basis of the EC spline space $\mathscr{S}$, called *the B-spline basis*. We shall now derive a number of properties of the B-spline basis already stated in [1]. This will clearly illustrate the efficiency of our approach which enables us to determine such properties in a very simple way.

Applying the Lyche–de Boor algorithm in order to calculate the value of the B-spline $\mathcal{N}_j$ at a given $x\in I$ from its poles will provide the support of this B-spline, i.e., the influence domain of the pole of index j. The poles of $\mathcal{N}_j$ being given by (6.10), we can conclude that

$$[t_i, t_{i+1}]\subset \operatorname{Supp} \mathcal{N}_j \Leftrightarrow j\in\{j_i-n, ..., j_i\},$$

to be compared with Lemma 6.7. Hence

$$\operatorname{Supp} \mathcal{N}_j=\mathscr{D}(X_j)=[x_j, x_{j+n+1}], \qquad j=-n, ..., m. \tag{6.11}$$

Moreover, the Lyche–de Boor algorithm also proves that $\mathcal{N}_j(x)>0$ for $x\in\,]x_j, x_{j+n+1}[$.

From the very definition of the B-spline basis, we can conclude that

$$S=\sum_{j=-n}^{m} s(X_j)\,\mathcal{N}_j \qquad \text{for all} \quad S\in\mathscr{S}. \tag{6.12}$$

In particular, for any $F\in\mathscr{U}\subset\mathscr{S}$, formula (6.12) gives

$$F=\sum_{j=-n}^{m} f(X_j)\,\mathcal{N}_j. \tag{6.13}$$

For instance, $\mathbb{1}=\sum_{j=-n}^{m}\mathcal{N}_j$. As an interesting example, we can apply formula (6.13) to $E(\cdot, y)\in\mathscr{U}$. On account of (5.8), this provides the following Marsden-type identity:

$$E(x, y)=\sum_{j=-n}^{m} \mathcal{N}_j(x)\,\Psi^*_{X_j}(y), \qquad x, y\in I. \tag{6.14}$$

Of course, in the same EC spline space $\mathscr{S}$, there are many such Marsden-type identities since there are different sequences N_l, $l=1, ..., q$ constructed by complementing the connection matrices B_l, $l=1, ..., q$. Such an identity has already been given by P. J. Barry *et al.* in the piecewise polynomial case, when all the multiplicities at the interior knots are equal to 1 [2].

7. FINAL REMARKS

Let $\mathcal{U}$ be a piecewise smooth W-space, $\mathcal{U}^*$ its dual space, and $[\cdot,\cdot]$ its associated canonical bilinear form. For $\mathbf{U}=(U^1, ..., U^d) \in \mathcal{U}^d$, and $U^* \in \mathcal{U}^*$, let us define $[\mathbf{U}, U^*] \in \mathbb{R}^d$ by

$$[\mathbf{U}, U^*] := ([U^1, U^*], ..., [U^d, U^*]). \tag{7.1}$$

For a given $a^\varepsilon \in I$, let us denote by $(U_0, ..., U_n)$ the basis of $\mathcal{U}$ characterized by

$$U_i^{(j)}(a^\varepsilon) = \delta_{ij}, \qquad i, j = 0, ..., n. \tag{7.2}$$

Let us fix $\Psi^* \in \mathcal{U}^*$. It was proved in Section 2 that

$$[U, \Psi^*] = \sum_{i=0}^{n} \lambda_i U^{(i)}(a^\varepsilon) \qquad \text{for all} \quad U \in \mathcal{U}. \tag{7.3}$$

Clearly, due to (7.2), the λ_i's are given by

$$\lambda_i = [U_i, \Psi^*], \qquad i = 0, ..., n. \tag{7.4}$$

From (7.1) and (7.3) we can derive

$$[\mathbf{U}, \Psi^*] = \sum_{i=0}^{n} \lambda_i \mathbf{U}^{(i)}(a^\varepsilon) \qquad \text{for all} \quad \mathbf{U} \in \mathcal{U}^d. \tag{7.5}$$

On the other hand, let us recall that *the osculating flat of order* i of $\mathbf{U} \in \mathcal{U}^d$ at a^ε, denoted by $\mathrm{Osc}_i\, \mathbf{U}(a^\varepsilon)$, is the affine flat going through the point $\mathbf{U}(a)$ and the direction of which is the linear space spanned by $\mathbf{U}'(a^\varepsilon), ..., \mathbf{U}^{(i)}(a^\varepsilon)$. Hence, on account of (7.4) and Corollary 2.12, we can state the following result.

LEMMA 7.1. *Given an integer* μ, $0 \leqslant \mu \leqslant n$, *and* $\Psi^* \in \mathcal{U}^*$, *the point* $[\mathbf{U}, \Psi^*]$ *belongs to* $\mathrm{Osc}_{n-\mu}\, \mathbf{U}(a^\varepsilon)$ *for all* $\mathbf{U} \in \mathcal{U}^d$ *iff* Ψ^* *vanishes on* $(a^\varepsilon)^\mu$ *and satisfies* $[U_0, \Psi^*] = 1$.

From now on, we suppose that $\mathcal{U}$ contains the constant functions. Then, whatever a and ε may be, with $a^\varepsilon \in I$, the function U_0 defined in (7.2) (i.e., the element of $\mathcal{U}$ which is characterized by $\Delta_n U_0(a^\varepsilon) = (1, 0, ..., 0)^T$) is $U_0 = \mathbb{1}$. As an immediate consequence of Lemma 7.1, we have:

PROPOSITION 7.2. *Let $\mathcal{T} \in I^n$ be a given n-tuple, such that* $\mathcal{T}^{ord} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$. *Then,* $\Psi^* \in \mathcal{U}^*$ *satisfies*

$$[\mathbf{U}, \Psi^*] \in \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i} \mathbf{U}(\tau_i^{\varepsilon_i}) \qquad \textit{for all} \quad \mathbf{U} \in \mathcal{U}^d$$

iff Ψ^* *vanishes on* $(\tau_i)^{\mu_i}$, $i = 1, ..., r$, *and satisfies the normalization condition* $[\mathbb{1}, \Psi^*] = 1$.

Let us come back to the assumptions of Section 5, which allows us to develop the blossoming principle in the space $\mathcal{U}$. Then, the blossom of a function $\mathbf{F} = (F^1, ..., F^d) \in \mathcal{U}^d$ will be defined as

$$\mathbf{f}(\mathcal{T}) := (f^1(\mathcal{T}), ..., f^d(\mathcal{T})), \tag{7.6}$$

i.e., by (7.1),

$$\mathbf{f}(\mathcal{T}) := [\mathbf{F}, \Psi^*_{\mathcal{T}}], \tag{7.7}$$

where $\Psi^*_{\mathcal{T}}$ vanishes on $\mathcal{T}$ and satisfies $[\mathbb{1}, \Psi^*_{\mathcal{T}}] = 1$. Thus, according to Proposition 7.2, if $\mathcal{T}^{ord} = (\tau_1^{\mu_1} \cdots \tau_r^{\mu_r})$,

$$\mathbf{f}(\mathcal{T}) \in \bigcap_{i=1}^{r} \mathrm{Osc}_{n-\mu_i} \mathbf{F}(\tau_i) \qquad \text{for all} \quad \mathbf{F} \in \mathcal{U}^d. \tag{7.8}$$

Under the same assumptions as in Section 5, it can be proved that, when $\mathbf{F}$ is nondegenerate (i.e., when the affine space spanned by the image of $\mathbf{F}$ is of dimension n), the right hand side of (7.8) consists of a single point [17, Sect. 6]. In the geometrical approach of the blossoming principle developed in [17], this point is specifically chosen to define the value of the blossom of such a nondegenerate function $\mathbf{F}$ at $\mathcal{T}$. Hence, the definition of the blossom through the duality principle presented here and the geometrical one presented in [17] actually lead to the same mathematical object.

REFERENCES

1. P. J. Barry, de Boor–Fix dual functionals and algorithms for Tchebycheffian B-spline curves, *Constr. Approx.* **12** (1996), 385–408.
2. P. J. Barry, N. Dyn, R. N. Goldman, and C. A. Mitchelli, Identities for piecewise polynomial spaces determined by connection matrices, *Aequationes Math.* **42** (1991), 123–136.
3. C. de Boor and G. Fix, Spline approximation by quasi-interpolants, *J. Approx. Theory* **8** (1973), 19–45.
4. N. Dyn and A. Ron, Recurrence relations for Tchebycheffian B-splines, *J. Analyse Math.* **51** (1988), 118–138.
5. R. N. Goldman, Dual polynomial bases, *J. Approx. Theory* **79** (1994), 311–346.

6. S. Karlin and W. J. Studden, "Tchebycheff Systems," Wiley–Interscience, New York, 1966.
7. S. Karlin and Z. Ziegler, Chebyshevian spline functions, *SIAM J. Numer. Anal.* **3** (1966), 514–543.
8. P. E. Koch and T. Lyche, Exponential B-splines in tension, *in* "Approximation Theory," pp. 361–364, Academic Press, New York, 1989.
9. P. E. Koch and T. Lyche, Construction of exponential tension B-splines of arbitrary order, *in* "Curves and Surfaces," pp. 255–258, Academic Press, Boston, 1991.
10. R. Kulkarni, P. J. Laurent, and M.-L. Mazure, Non affine blossoms and subdivision for Q-splines, *in* "Mathematical Methods in CAGD and Image Processing," pp. 367–380, Academic Press, New York, 1992.
11. E. T. Y. Lee, Marsden's identity, *Comput. Aided Geom. Design* **13** (1996), 287–305.
12. T. Lyche, A recurrence relation for Chebyshevian B-splines, *Constr. Approx.* **1** (1985), 155–173.
13. M. J. Marsden, An identity for spline functions with applications to variation-diminishing spline approximation, *J. Approx. Theory* **3** (1970), 7–49.
14. M.-L. Mazure, Blossoming of Chebyshev splines, *in* "Mathematical Methods for Curves and Surfaces," pp. 355–364, Vanderbilt Univ. Press, 1995.
15. M.-L. Mazure, "Chebyshev Spaces," RR 952M IMAG, Université Joseph Fourier, Grenoble, 1996.
16. M.-L. Mazure, "Chebyshev Blossoming," RR 953M IMAG, Université Joseph Fourier, Grenoble, 1996.
17. M.-L. Mazure, Blossoming: A geometric approach, *Constr. Approx.* **15** (1999), 33–68.
18. M.-L. Mazure and P. J. Laurent, Affine and non affine blossoms, *in* "Computational Geometry," pp. 201–230, World Scientific, Singapore, 1993.
19. M.-L. Mazure and P. J. Laurent, Marsden identities, Blossoming and de Boor–Fix formula, *in* "Advanced Topics in Multivariate Approximation," pp. 227–242, World Scientific, Singapore, 1997.
20. M.-L. Mazure and P. J. Laurent, Nested sequences of Chebyshev spaces and shape parameters, *Math. Modelling Numer. Anal.* **32** (1998), 773–788.
21. M.-L. Mazure and H. Pottmann, Tchebycheff curves, *in* "Total Positivity and Its Applications," pp. 187–218, Kluwer Academic, Dordrecht, 1996.
22. H. Pottmann, The geometry of Tchebysheffian splines, *Comput. Aided Geom. Design* **10** (1993), 181–210.
23. H. Pottmann and M. G. Wagner, Helix splines as an example of affine Tchebycheffian splines, *Adv. Comput. Math.* **2** (1994), 123–142.
24. L. Ramshaw, "Blossoming: A Connect-the Dots Approach to Splines," Tech. Rep. Digital Systems Research Center, Palo Alto, CA, 1987.
25. L. Ramshaw, Blossoms are polar forms, *Comput. Aided Geom. Design* **6** (1989), 323–358.
26. L. L. Schumaker, "Spline Functions," Wiley–Interscience, New York, 1981.
27. H.-P. Seidel, New algorithms and techniques for computing with geometrically continuous spline curves of arbitrary degree, *Math. Modelling Numer. Anal.* **26** (1992), 149–176.
28. Y. Stefanus and R. N. Goldman, Blossoming Marsden's identity, *Comput. Aided Geom. Design* **9** (1992), 73–84.
29. M. G. Wagner and H. Pottmann, Symmetric Tchebysheffian B-splines schemes, *in* "Curves and Surfaces in Geometric Design," pp. 483–490, A. K. Peters, 1994.